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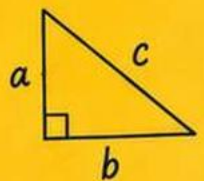
ESSENTIAL FORMULAS FOR ACT MATH SUCCESS

ACT MATH

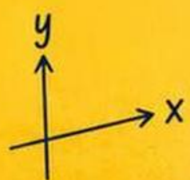
★ FORMULA REVIEW ★



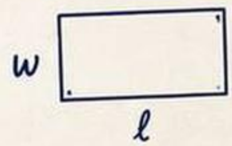
$a^2 + b^2 = c^2$

$y = mx + b$

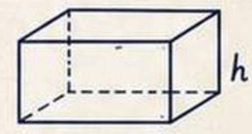
$A = \pi r^2$



$A = l \times w$



$V = lwh$



$P = 2l + 2w$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

KEY FORMULAS AT A GLANCE | QUICK REFERENCE ANYTIME | FOCUS ON WHAT MATTERS | IMPROVE ACCURACY AND SPEED | CONFIDENCE ON TEST DAY

- ✓ Algebra
- ✓ Geometry
- ✓ Trigonometry
- ✓ Coordinate Geometry
- ✓ Statistics & Data
- ✓ And More!

Review. Practice.
Achieve Your Best.

ACE THE ACT WITH CONFIDENCE | MANAGE TIME EFFECTIVELY | RAISE YOUR SCORE AND OPPORTUNITIES | YOUR FUTURE. YOUR SUCCESS.

1 Whole Numbers, Properties & Order

Number rules

Place value	A digit's value depends on its position: ones, tens, hundreds, tenths, hundredths, and so on.
Order of operations	Parentheses, exponents, multiply or divide left to right, then add or subtract left to right.
Commutative property	$a+b=b+a$ and $ab=ba$.
Associative property	$(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$.
Distributive property	$a(b+c)=ab+ac$.
Identity properties	$a+0=a$ and $a\cdot 1=a$.
Inverse properties	$a+(-a)=0$ and $a\cdot\frac{1}{a}=1$ when $a\neq 0$.

Tutor's Note

Order of operations keeps everyone evaluating expressions the same way. Multiplication and division are a team: do whichever appears first from left to right. Addition and subtraction work the same way.

Example

$18-3(2+4)^2\div 6$. Parentheses: 6. Exponent: 36. Multiply/divide: $3\cdot 36\div 6=18$. Then $18-18=0$.

PEMDAS

PEMDAS does not mean all multiplication before all division. Work multiplication and division left to right.

Order of operations (PEMDAS) Work top to bottom, but multiply/divide share a step (left to right) and so do add/subtract.

1. Parentheses ( ) → 2. Exponents x^2 → 3. $\times$ $\div$ left→right → 4. $+$ $-$ left→right

2 Factors, Multiples & Number Types

Factors and number sense

Factor	A whole number that divides another whole number evenly.
Multiple	A result of multiplying a number by a whole number.
Prime number	A whole number greater than 1 with exactly two factors: 1 and itself.
Composite number	A whole number greater than 1 with more than two factors.
GCF	Greatest common factor: the largest factor shared by numbers.
LCM	Least common multiple: the smallest positive multiple shared by numbers.
GCF-LCM connection	For positive integers a, b: $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.
Divisibility: $2, 5, 10$	By 2 if even; by 5 if ending in 0 or 5; by 10 if ending in 0.
Divisibility: $3, 4, 6, 9$	By 3 or 9 if the digit sum is divisible by 3 or 9; by 4 if the last two digits form a multiple of 4; by 6 if divisible by 2 and 3.

Tutor's Note

GCF helps you simplify and factor. LCM helps you build common denominators. Prime factorization is the cleanest way to find both.

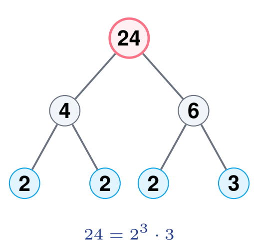
Example

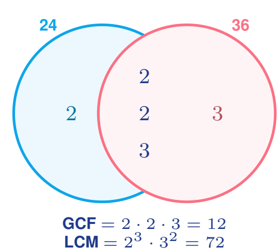
$24 = 2^3 \cdot 3$ and $36 = 2^2 \cdot 3^2$. GCF $= 2^2 \cdot 3 = 12$ and LCM $= 2^3 \cdot 3^2 = 72$.

ZERO

Zero is a multiple of every nonzero whole number, but it is not used as the least positive common multiple.

Prime factor trees & the GCF/LCM Venn Break each number into primes, then share them: the overlap is the GCF and the whole picture is the LCM.





3 Fractions & Mixed Numbers

Fraction operations

Equivalent fractions	$\frac{a}{b} = \frac{ak}{bk}$ when $b \neq 0$ and $k \neq 0$.
Simplify	$\frac{a}{b}$ is simplified by dividing numerator and denominator by their GCF, with $b \neq 0$.
Add/subtract	$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, with $b, d \neq 0$.
Multiply	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, with $b, d \neq 0$.
Divide	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, with $b, c, d \neq 0$.
Reciprocal	The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$, with $a, b \neq 0$.
Mixed to improper	$a\frac{b}{c} = \frac{ac + b}{c}$ for positive mixed numbers; keep a negative sign outside until the end.
Part of a whole	A fraction $\frac{a}{b}$ means a parts out of b equal parts.

Tutor's Note

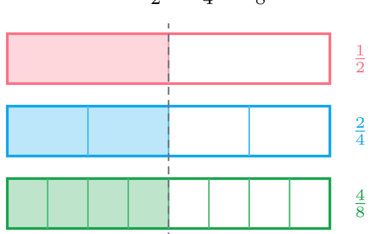
To add or subtract fractions, the denominators must name the same size pieces. To multiply, go straight across. To divide, keep the first fraction, change division to multiplication, and flip the second fraction.

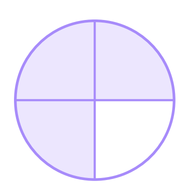
Example

$$\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{9}{6} = \frac{3}{2} = 1\frac{1}{2}.$$

DENOMINATORS — A denominator can never be 0. Division by zero is undefined.

Fraction models: equivalent fractions The same amount can be cut into more, smaller pieces. Each bar below is shaded *half*, so $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$.





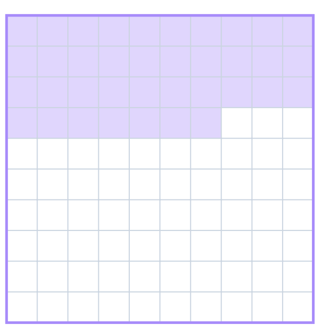
$\frac{3}{4}$ shaded

4 Decimals, Percents & Money Math

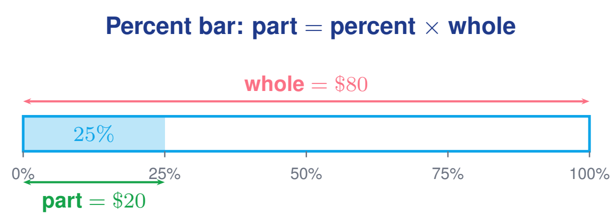
Decimal and percent formulas

Add/subtract decimals	Line up the decimal points, then add or subtract as usual.
Multiply decimals	Multiply as whole numbers, then place the point so the answer has the total number of decimal places.
Divide decimals	Move both decimals right until the divisor is whole, then divide.
Decimal to percent	Move the decimal two places right: $0.37 = 37\%$.
Percent to decimal	Move the decimal two places left: $85\% = 0.85$.
Percent to fraction	$p\% = \frac{p}{100}$, then simplify if possible.
Percent equation	part = percent · whole, using percent as a decimal.
Percent change	$\frac{\text{new} - \text{old}}{\text{old}} \cdot 100\%$, where old $\neq 0$.
Simple interest	$I = Prt$ and $A = P + I = P(1 + rt)$.
Tax / tip / markup	Total = original amount $\times(1 + r)$, where r is the rate as a decimal.
Discount / markdown	Sale price = original price $\times(1 - r)$, where r is the discount rate.

Percent means "out of 100" Shade a 10×10 grid: each little square is 1%. The same value is a fraction over 100 and a decimal.



$37\% = 0.37 = \frac{37}{100}$



$0.25 \times 80 = 20$

Tutor's Note

The most common percent mistake is choosing the wrong whole. The whole is the original or total amount that the percent is based on.

Example

A jacket costs $\$80$ and is discounted 25%. Discount $= 0.25(80) = 20$, so the sale price is $\$60$.

MONEY

Round money to the nearest cent only at the end unless the problem tells you to round earlier.

5 Ratios, Rates & Proportions

Proportion tools

Ratio	$a : b$, a to b, and $\frac{a}{b}$ all compare two quantities.
Rate	A ratio with different units, such as miles per hour.
Unit rate	A rate with denominator 1.
Proportion	$\frac{a}{b} = \frac{c}{d}$ means $ad = bc$, with $b, d \neq 0$.
Scale factor	scale factor $= \frac{\text{new length}}{\text{original length}}$.
Similar figures	Corresponding side lengths are proportional and corresponding angles match.
Constant of proportionality	$y = kx$, where $k = \frac{y}{x}$ when $x \neq 0$.
Proportional graph	A proportional relationship graphs as a straight line through $(0, 0)$.
Unit price	unit price $= \frac{\text{total cost}}{\text{number of units}}$.
Distance, rate, time	$d = rt$; so $r = \frac{d}{t}$ and $t = \frac{d}{r}$.
Percent proportion	$\frac{\text{part}}{\text{whole}} = \frac{\text{percent}}{100}$, with whole $\neq 0$.

Tutor's Note

A proportion is two equal ratios. Cross-multiplication works because multiplying both sides by the denominators clears the fractions.

Example

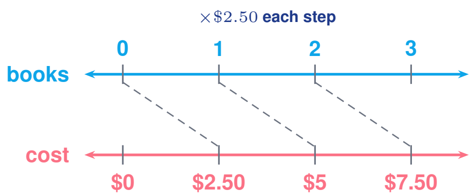
If 3 notebooks cost $\$7.50$, the unit rate is $7.50 \div 3 = \$2.50$ per notebook.

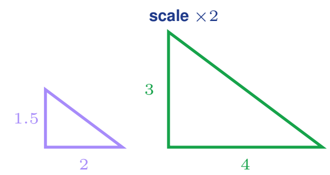
UNITS

Ratios compare. Rates compare with units. Keep units attached until the answer makes sense.

Ratio tables & similar figures

Scale both quantities by the same number and the ratio stays the same.





same shape, sides $\times 2$

6 Integers & Absolute Value

Signed number rules

Add same signs	Add absolute values and keep the common sign.
Add different signs	Subtract absolute values and keep the sign of the number farther from 0.
Subtract	$a - b = a + (-b)$.
Multiply/divide signs	Same signs give positive; different signs give negative.
Zero pairs	$a + (-a) = 0$; opposites cancel in addition.
Absolute value	$\lvert a \rvert$ is the distance from 0, so $\lvert a \rvert \geq 0$.
Opposites	a and $-a$ are the same distance from 0 in opposite directions.
Coordinate distance on a line	Distance between a and b is $\lvert a - b \rvert$.

Tutor's Note

For multiplying or dividing, count negatives: an even number of negative signs gives a positive result; an odd number gives a negative result.

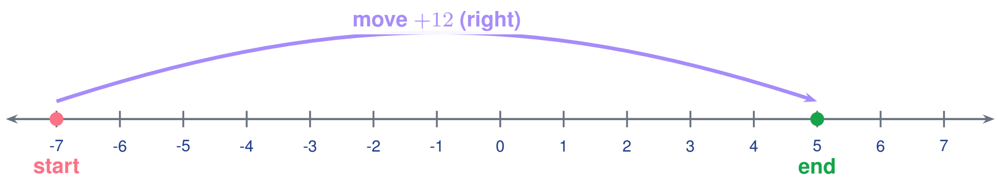
Example

$-7 + 12 = 5$ because the signs are different, so subtract $12 - 7$ and keep the sign of 12.

SIGNS

Do not treat subtraction like multiplication. For $5 - (-3)$, add the opposite: $5 + 3 = 8$.

Adding integers on a number line Start at the first number, then *move*: right to add a positive, left to add a negative.



$-7+12=5.$ The two points are $|-7-5|=12$ apart.

7 Expressions, Equations & Inequalities

Algebra basics

Expression	Numbers, variables, and operations without an equals sign.
Equation	A statement that two expressions are equal.
Evaluate	Substitute a number for a variable and simplify.
Substitution	Replace a variable with a given value, then use order of operations.
Combine like terms	Add/subtract coefficients of terms with the same variable part.
Distribute	$a(b+c)=ab+ac.$
One-step equation	$x+a=b \Rightarrow x=b-a$ and $ax=b \Rightarrow x=\frac{b}{a}$ when $a \neq 0.$
Two-step equation	For $ax+b=c$, subtract b first, then divide by a when $a \neq 0.$
Inequality rule	If multiplying or dividing by a negative number, flip the inequality symbol.

Tutor's Note

Solving means keeping the equation balanced while isolating the variable. Whatever you do to one side, do to the other side.

Example

Solve $3x+5=20$. Subtract 5: $3x=15$. Divide by 3: $x=5$.

9 Coordinate Plane & Linear Patterns

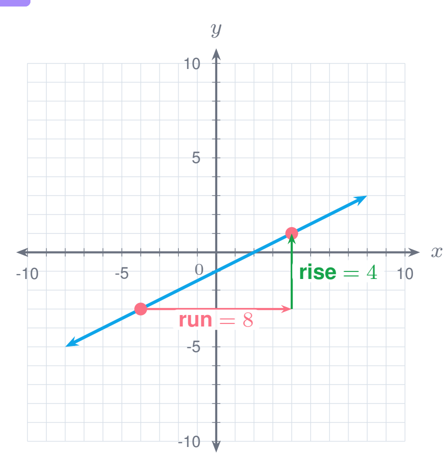
Graphing basics

Ordered pair	(x, y) gives horizontal movement first, vertical movement second.
Quadrants	I: $(+,+)$, II: $(-,+)$, III: $(-,-)$, IV: $(+,-)$.
Slope	$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$, with $x_2 \neq x_1$.
Rate of change	Another name for slope: how much y changes for each 1 unit of x.
y-intercept	The point where a graph crosses the y-axis.
x-intercept	The point where a graph crosses the x-axis.
Linear pattern	$y = mx + b$, where m is slope and b is the y-intercept.
Direct variation	$y = kx$ is a proportional linear pattern through the origin.
Horizontal line	$y = c$ has slope 0.
Vertical line	$x = c$ has undefined slope.

Tutor's Note

The coordinate plane is a map. The x-axis moves left and right; the y-axis moves down and up. Slope tells how much y changes for each 1 step in x.

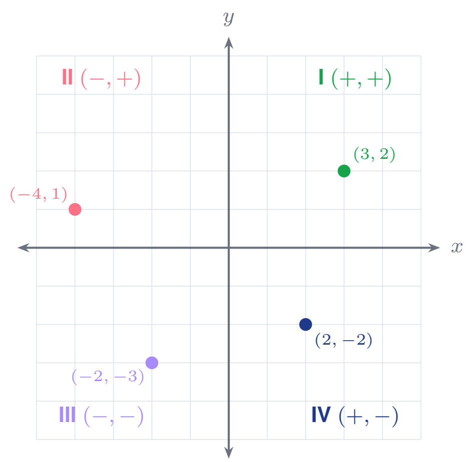
Visual: slope is rise over run



GRAPH | A vertical line fails the function test because one input has many outputs.

The coordinate plane: quadrants & plotting

An ordered pair (x, y) moves right/left first, then up/down. The signs tell you the quadrant.



Graphing checkpoint

Read the plane like a map: horizontal first, vertical second.

Ordered pair
$(-4, 3)$ means left 4, then up 3. The x-move comes first.

Quadrant signs
I $(+,+)$, II $(-,+)$, III $(-,-)$, IV $(+,-)$.

Slope
$m = \frac{\text{rise}}{\text{run}}$. Up/right is positive; down/right is negative.

Intercepts
y-intercept: where the graph crosses the y-axis. x-intercept: where it crosses the x-axis.

Line patterns
$y = mx + b$ shows slope m and starting value b.

Fast check
A vertical line has undefined slope because its run is 0.

10 Geometry & Measurement

Plane geometry

Rectangle perimeter	$P = 2l + 2w.$
Rectangle area	$A = lw.$
Square perimeter / area	$P = 4s$ and $A = s^2.$
Triangle area	$A = \frac{1}{2}bh.$
Parallelogram area	$A = bh.$
Trapezoid area	$A = \frac{1}{2}(b_1 + b_2)h.$
Circle circumference	$C = 2\pi r = \pi d.$
Circle area	$A = \pi r^2.$
Pythagorean theorem	$a^2 + b^2 = c^2$ for right triangles, where c is the hypotenuse.

Area formulas you can see Area is the flat space inside a figure; the height is always perpendicular to the base.

Rectangle $A = bh$ | Triangle $A = \frac{1}{2}bh$ | Parallelogram $A = bh$ | Circle $A = \pi r^2$

Geometry checkpoint Name the measurement before choosing a formula.

Perimeter
Add side lengths around the outside. Units stay linear.

Area
Covering a flat region uses square units. Rectangle: $A = lw$.

Triangle area
$A = \frac{1}{2}bh$. The height must be perpendicular to the base.

Circles
$C = 2\pi r = \pi d$ and $A = \pi r^2$. Diameter is twice the radius.

Pythagorean check
Use $a^2 + b^2 = c^2$ only for right triangles; c is the hypotenuse.

Fast check
Area and volume answers should include square or cubic units.

Solid geometry: volume & surface area

Rectangular prism	$V = lwh$; surface area $SA = 2lw + 2lh + 2wh$.
Cube	$V = s^3$; surface area $SA = 6s^2$.
Cylinder	$V = \pi r^2 h$; surface area $SA = 2\pi r^2 + 2\pi rh$.
Triangular prism	$V = \left(\frac{1}{2}bh\right) L$; $SA =$ two triangle ends + three rectangle faces.
Cone	$V = \dfrac{1}{3}\pi r^2 h$.
Pyramid	$V = \dfrac{1}{3}(\text{base area})\, h$.
Sphere	$V = \dfrac{4}{3}\pi r^3$; surface area $SA = 4\pi r^2$.
Surface area idea	Add the areas of every outside face or curved surface; nets help organize the faces.

Tutor's Note

Area is measured in square units because it covers a flat region. Volume is measured in cubic units because it fills space. Make sure all measurements use the same unit before using a formula.

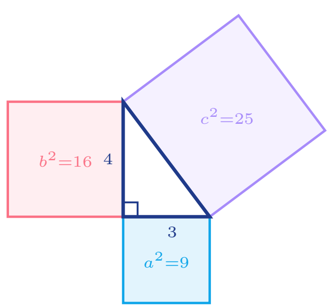
Example

A triangle with base 10 cm and height 7 cm has area $A = \frac{1}{2}(10)(7) = 35$ square centimeters.

GEOMETRY

For the Pythagorean theorem, c is always the longest side, across from the right angle.

The Pythagorean theorem: $a^2 + b^2 = c^2$ For a right triangle, the square built on the hypotenuse equals the two squares on the legs added together.



The legs are $a = 3$ and $b = 4$, so
$c^2 = a^2 + b^2 = 9 + 16 = 25$,
which gives $c = \sqrt{25} = 5$.
The hypotenuse c is always the longest side, opposite the right angle.

11 Measurement & Unit Conversions

Customary (U.S.) units

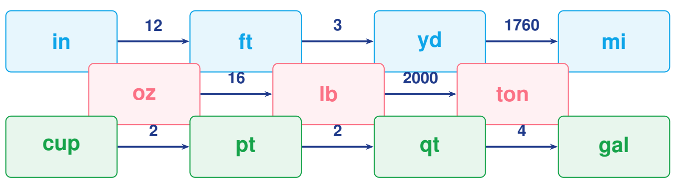
Length 12 in $= 1$ ft; 3 ft $= 1$ yd; 5280 ft $= 1$ mi.

Weight 16 oz $= 1$ lb; 2000 lb $= 1$ ton.

Capacity 8 fl oz $= 1$ cup; 2 cups $= 1$ pt; 2 pt $= 1$ qt; 4 qt $= 1$ gal.

Unit conversion checkpoint

Write the unit you want to cancel on the bottom.



Length path
1 yd $= 3$ ft and 1 ft $= 12$ in, so 2 yd $= 72$ in.

Weight path
1 lb $= 16$ oz; multiply pounds by 16 to get ounces.

Capacity path
1 gal $= 4$ qt $= 8$ pt $= 16$ cups.

Fraction setup
5 ft $\times \frac{12 \text{ in}}{1 \text{ ft}}$: feet cancel.

Reasonableness
Smaller units make larger numbers; larger units make smaller numbers.

Fast check
Keep units attached until the final answer. They show whether the setup is right.

Try it 4 ft $= 48$ in 3 lb $= 48$ oz 2 gal $= 8$ qt $= 32$ cups Smaller units make the number larger.

Metric units & time

Metric length 10 mm $= 1$ cm; 100 cm $= 1$ m; 1000 m $= 1$ km.

Metric mass 1000 mg $= 1$ g; 1000 g $= 1$ kg.

Metric capacity 1000 mL $= 1$ L.

Metric prefixes kilo $= 1000$, hecto $= 100$, deca $= 10$, deci $= \frac{1}{10}$, centi $= \frac{1}{100}$, milli $= \frac{1}{1000}$.

Time 60 s $= 1$ min; 60 min $= 1$ hr; 24 hr $= 1$ day; 7 days $= 1$ wk.

Temperature $F = \frac{9}{5}C + 32$ and $C = \frac{5}{9}(F - 32)$.

Converting units: multiply by a fraction equal to 1 Write the conversion so the *old* unit cancels and the *new* unit is left.

$$5 \text{ ft} \times \frac{12 \text{ in}}{1 \text{ ft}} = 60 \text{ in}$$

the "ft" units cancel, leaving inches

Tutor's Note

To convert, multiply by a fraction that equals 1 (the same amount written two ways). Put the unit you want to cancel on the bottom and the unit you want to keep on top.

Example

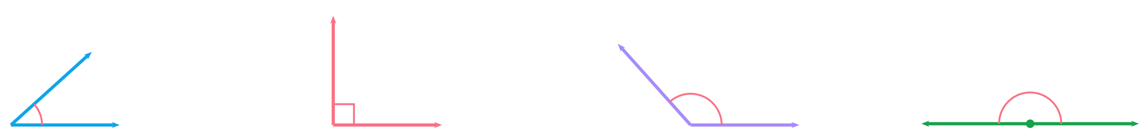
Convert 3 yards to inches: $3 \text{ yd} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in}}{1 \text{ ft}} = 108$ in.

UNITS

Metric prefixes move the decimal by powers of 10: from km to m multiply by 1000; from m to cm multiply by 100.

12 Angles, Lines & Polygons

Types of angles Angles are measured in degrees. A right angle is exactly $90°$; a straight angle is $180°$.



Acute — less than $90°$

Right — exactly $90°$

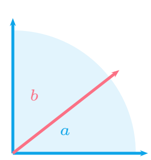
Obtuse — between $90°$ and $180°$

Straight — exactly $180°$

Angle & figure facts

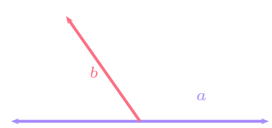
Right / straight angle	A right angle is $90°$; a straight angle is $180°$; a full turn is $360°$.
Complementary angles	Two angles whose measures add to $90°$.
Supplementary angles	Two angles whose measures add to $180°$ (a straight line).
Vertical angles	When two lines cross, opposite angles are equal.
Triangle angle sum	The three interior angles of a triangle add to $180°$.
Quadrilateral angle sum	The four interior angles of a quadrilateral add to $360°$.
Polygon angle sum	For n sides, interior angles add to $(n-2)\cdot 180°$.
Regular polygon angle	Each interior angle is $\frac{(n-2)180°}{n}$.
Exterior angles	One exterior angle at each vertex adds to $360°$; regular exterior angle is $\frac{360°}{n}$.
Parallel lines	Lines in a plane that never meet; a crossing line makes equal matching angles.

Angle pairs Complementary angles add to $90°$, supplementary angles add to $180°$, and vertical angles (across an X) are equal.



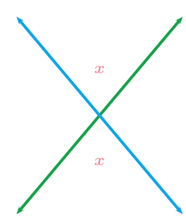
Complementary

$a + b = 90°$



Supplementary

$a + b = 180°$



Vertical

opposite angles equal

Tutor's Note

Most angle problems are just addition. If two angles form a right angle, they are complementary; if they form a straight line, they are supplementary. Set the known sum equal to the parts and solve.

Example

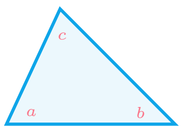
If one angle of a complementary pair is $35°$, the other is $90° - 35° = 55°$.

ANGLES

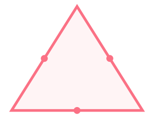
The angles of a triangle always total $180°$, so two angles determine the third: $c = 180° - a - b$.

Triangle angle sum & figure types

The three angles of any triangle add to $180°$; the four angles of any quadrilateral add to $360°$.



$a + b + c = 180°$



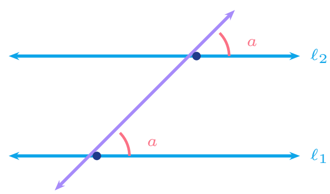
equilateral: all sides =

rectangle

4 right angles

Parallel lines cut by a transversal

A line crossing two parallel lines makes equal matching angles.



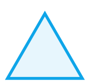
Corresponding angles (same position at each crossing) are **equal**.

Alternate interior angles (opposite sides, between the lines) are **equal**.

Co-interior angles (same side, between the lines) are **supplementary** — they add to $180°$.

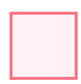
Interior angle sums: $(n - 2) \times 180°$

Every extra side adds another $180°$ to the total.



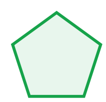
Triangle

3 sides = $180°$



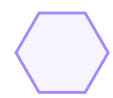
Quadrilateral

4 sides = $360°$



Pentagon

5 sides = $540°$



Hexagon

6 sides = $720°$

Angle checkpoint

Most angle questions are sum facts plus simple equations.

Pairs
$a + b = 90°$ for complementary;
$a + b = 180°$ for supplementary.

Triangles
$a + b + c = 180°$. Two angles determine the third.

Quadrilaterals
Four interior angles add to $360°$.

Polygons
Interior sum $= (n - 2)180°$. Divide by n only for regular polygons.

Parallel lines
Corresponding and alternate interior angles match; same-side interior angles add to $180°$.

Fast check
A straight line is $180°$ and a full turn is $360°$.

13 Data, Statistics & Probability

Statistics formulas

Mean	$\bar{x} = \dfrac{\text{sum of values}}{\text{number of values}}$.
Median	The middle value after the data are ordered.
Mode	The value or values that occur most often.
Range	Maximum minus minimum.
Quartiles	Q_1, median, Q_3 split ordered data into four equal parts.
Interquartile range	IQR $= Q_3 - Q_1$, the spread of the middle half.
Mean absolute deviation	MAD $= \dfrac{\sum \lvert x - \bar{x} \rvert}{n}$, the average distance from the mean.
Probability	$P(\text{event}) = \dfrac{\text{favorable outcomes}}{\text{total equally likely outcomes}}$.
Experimental probability	$\dfrac{\text{times event occurs}}{\text{number of trials}}$.
Complement	$P(\text{not } A) = 1 - P(A)$.
Simple counting principle	If one choice has m options and another has n options, together there are mn outcomes.

Tutor's Note

Statistics summarize data. Probability predicts chance. Both depend on reading the question carefully: are the outcomes equally likely, and what exactly counts as a success?

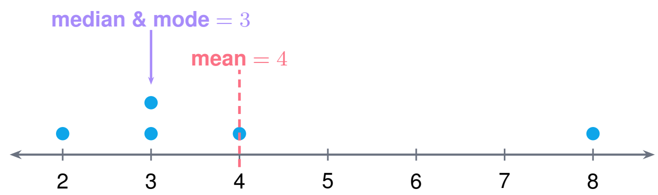
Example

For data $4, 5, 6, 9$, the mean is $\frac{4+5+6+9}{4} = 6$ and the range is $9 - 4 = 5$.

DATA

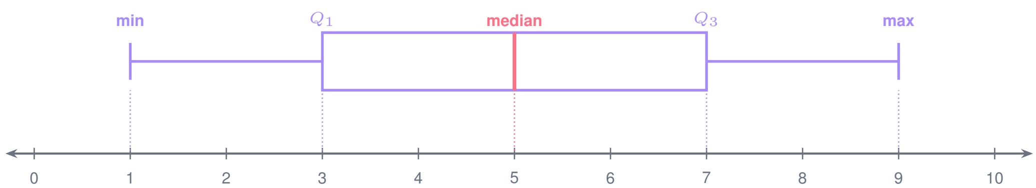
Outliers can pull the mean more than the median. If one value is far away from the rest, compare both measures.

Mean, median & mode on a dot plot Stack a dot for each value, then read the center three ways. The mean balances the data; the median is the middle; the mode is the most common.



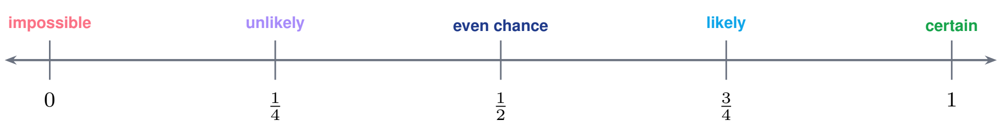
Data $2, 3, 3, 4, 8$: mean $\frac{20}{5} = 4$, median 3, mode 3, range $8 - 2 = 6$.

The five-number summary & box plot Order the data, then mark the minimum, first quartile Q_1, median, third quartile Q_3, and maximum.



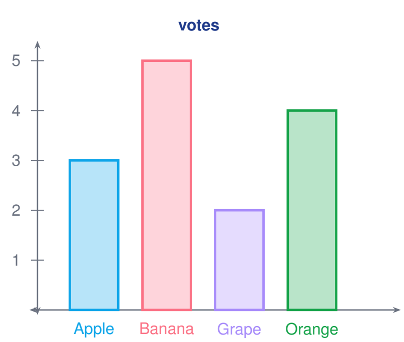
The box holds the middle half of the data; its width $(Q_3 - Q_1)$ is the interquartile range (IQR).

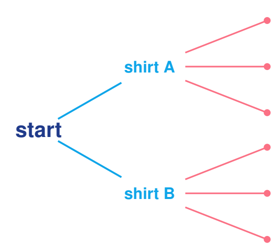
The probability scale Every probability is a number from 0 (impossible) to 1 (certain). Half-way is an even chance.



0% to 100% — e.g. one head on a fair coin flip is $\frac{1}{2} = 50\%$.

Reading data & counting outcomes A bar graph compares categories; the counting principle multiplies the choices at each stage.





$2 \times 3 = 6$ outfits

1 Foundations of Algebra

Before we build anything tall, we need a solid base. This section covers the ground rules that every later topic leans on — how numbers behave, the order we work in, and how to handle signs and fractions without fear.

The Real-Number Properties

These four properties are the "rules of the game." You already use them without thinking; algebra just gives them names.

Commutative (order)

$a + b = b + a$

$a \cdot b = b \cdot a$

Associative (grouping)

$(a + b) + c = a + (b + c)$

$(ab)c = a(bc)$

Distributive (share out)

$a(b + c) = ab + ac$

Identity & Inverse

$a + 0 = a, \quad a \cdot 1 = a$

$a + (-a) = 0, \quad a \cdot \frac{1}{a} = 1 \ (a \neq 0)$

Tutor's Note

Commutative means order doesn't matter for adding or multiplying: $3 + 5$ and $5 + 3$ both give 8. **Associative** means it doesn't matter how you group them: you can add the first two numbers or the last two and land in the same place. The **distributive** property is the real workhorse of algebra — it lets you "hand out" a multiplier to everything inside parentheses, which is how we expand and factor later. Notice these work for $+$ and $\times$, but *not* for subtraction or division: $6 - 2 \neq 2 - 6$.

Order of Operations (PEMDAS)

When an expression has several operations, we all need to agree on what to do first — otherwise the same expression could give different answers.

Do them in this order

Parentheses $\rightarrow$ **E**xponents $\rightarrow$ **M**ultiplication & **D**ivision (left $\rightarrow$ right) $\rightarrow$ **A**ddition & **S**ubtraction (left $\rightarrow$ right)

Tutor's Note

A useful phrase is "*Please Excuse My Dear Aunt Sally*." The one trap: multiply and divide are a *team* — do whichever comes first reading left to right, not all multiplication before all division. Same goes for add and subtract.

Example

$3 + 4 \times 2^2 - (6 - 2)$. First parentheses: $(6 - 2) = 4$. Then the exponent: $2^2 = 4$. Then multiply: $4 \times 4 = 16$. Finally left to right: $3 + 16 - 4 = 15$.

Solving Linear Equations

The goal: isolate the variable

Undo operations in reverse PEMDAS order.

$$ax + b = c \implies x = \frac{c-b}{a} \quad (a \neq 0)$$

Tutor's Note

Picture a wrapped present: it was wrapped in a certain order, and you unwrap it in the *opposite* order. If x was multiplied by a and then had b added, you undo it by first subtracting b, then dividing by a. Every step you do to the left side, do to the right side too — that keeps the scale balanced.

Example

$3x + 5 = 20$. Subtract 5: $3x = 15$. Divide by 3: $x = 5$. Check: $3(5) + 5 = 20$. ✓

Special Solution Cases

Sometimes the variable disappears as you solve. That's not a mistake — it's telling you something.

One solution

$2x + 1 = x + 4 \Rightarrow x = 3$

Infinitely many

$2(x + 1) = 2x + 2$ (always true)

No solution

$x + 2 = x + 5 \Rightarrow 2 = 5$ (false)

Literal equations

$A = \ell w \Rightarrow w = \frac{A}{\ell} \quad (\ell \neq 0)$

Tutor's Note

If you end with something *false* like $2 = 5$, no number can make it work, so there's *no solution*. If you end with something *always true* like $2 = 2$, then *every* number works. A "literal" equation just has more than one letter — you solve for one variable using the same balancing moves, treating the others like numbers.

Inequalities

The one new rule

Solve exactly like an equation, **but flip the inequality sign** whenever you multiply or divide both sides by a *negative* number.

$$-2x < 6 \implies x > -3$$

Tutor's Note

Why the flip? Think about a true statement like $3 < 5$. Multiply both sides by -1 and you get -3 and -5. But -3 is actually *greater* than -5, so the sign has to turn around to stay true. That's the whole reason — negatives reverse order on the number line.

Visual: graphing $x > -3$

Reading the symbols

$<$ less than $>$ greater than
$\leq$ at most $\geq$ at least

Compound inequalities

AND: $a < x < b$ (between)
OR: $x < a$ or $x > b$

GRAPH — A solid dot or closed bracket means "or equal to" ($\leq, \geq$). An open dot means strictly less or strictly greater.

Absolute-Value Equations & Inequalities

Split into two cases

$$|ax + b| = c \Rightarrow ax + b = c \text{ or } ax + b = -c \quad (c \geq 0)$$

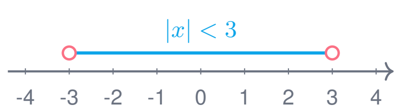
$$|x| < c \Rightarrow -c < x < c \qquad |x| > c \Rightarrow x < -c \text{ or } x > c \quad (c > 0)$$

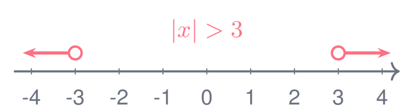
Tutor's Note

Because absolute value measures distance, $|x| = 5$ asks "what is 5 away from zero?" — and there are *two* answers, 5 and -5. So every absolute-value equation splits into two. For "less than," you want the points *close* to zero (a sandwich, between two values); for "greater than," you want the points *far* from zero (two pieces heading outward).

ABS — If an absolute-value equation equals a negative number, it has no solution. For absolute-value inequalities, check $c < 0$ as a special case because distance is never negative.

Visual: close to zero vs. far from zero





Translating Words into Algebra

Half of algebra is turning English into symbols. Here's your dictionary.

- sum / increased by / more than $\to +$
- difference / less than / fewer $\to -$
- product / of / times / twice $\to \times$
- quotient / per / split $\to \div$
- is / was / equals / gives $\to =$
- a number / unknown $\to x$

WORDS "5 less than a number" is $x - 5$, *not* $5 - x$. The order matters because "less than" reverses the order in the expression.

3 Functions, Lines & Slope

A function is one of the biggest ideas in all of math: a reliable machine that turns each input into exactly one output. Lines are the simplest functions, and slope is the number that describes how steep they are.

What a Function Is

Definition

A **function** gives exactly one output y for each input x. We write $y = f(x)$. **Vertical Line Test:** a graph is a function if no vertical line ever hits it twice.

Tutor's Note

Think of a vending machine: press B4 and you always get the same snack. That "same input, same single output" rule is what makes something a function. The notation $f(x)$ is read "f of x" and just means "the output when the input is x." It is *not* multiplication — it's a name tag for the machine's result.

Visual: vertical line test

How to read it

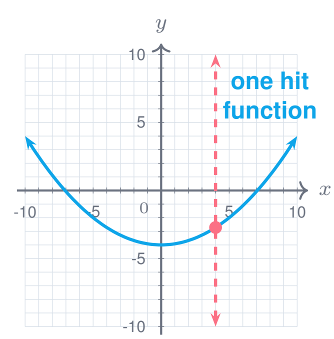
A graph is a function when each input x has only one output y.

The dashed vertical line touches this curve once, so it passes the test.

Domain — allowed inputs.

Range — possible outputs.

Evaluate: $f(3) = 2(3) + 1 = 7$.



Scan Me

Slope — Steepness in a Number

Slope between two points

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} \quad (x_2 \neq x_1)$$

Tutor's Note

Slope answers "for every step I take to the right, how far do I go up or down?" That's the *rise over run*. A big slope is a steep hill; a small slope is a gentle ramp. The sign tells direction: positive climbs as you read left to right, negative descends.

SLOPE

Subtract the x's and the y's in the *same order*. Mixing the order flips the sign and changes the slope.

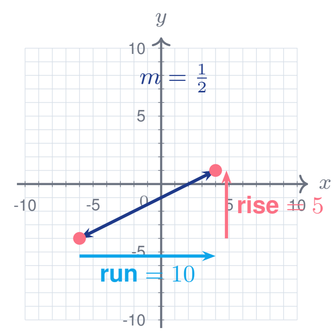
Visual: rise over run

Slope is a rate

Count the horizontal change first, then the vertical change.

Here, run $= 10$ and rise $= 5$, so $m = \frac{5}{10} = \frac{1}{2}$.

Positive slope rises left to right. Negative slope falls left to right.



- $m > 0$: line rises (↗)
- $m < 0$: line falls (↘)
- $m = 0$: flat, horizontal $y = b$
- m undefined: vertical $x = a$

Forms of a Linear Equation

The same line can be written several ways. Each form is handy for a different job.

Slope-Intercept
$y = mx + b$
slope m, y-intercept b

Standard
$Ax + By = C$
(A and B not both 0)

Point-Slope
$y - y_1 = m(x - x_1)$

Intercepts
x-int: set $y = 0$
y-int: set $x = 0$

Tutor's Note

Use **slope-intercept** when you know the slope and where the line crosses the y-axis — you can graph it instantly. Use **point-slope** when you know the slope and *any* one point; it's the fastest way to write a line's equation from scratch. **Standard form** is tidy for finding both intercepts. They all describe the same line — just dressed for different occasions.

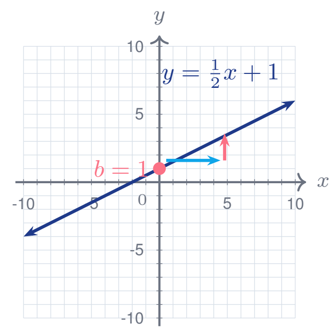
Visual: slope-intercept form

Use slope-intercept form

The line crosses the y-axis at $b = 1$.

The slope is $\frac{1}{2}$: move right 2, then up 1; or right 4, then up 2.

The arrows show the line continues forever in both directions.



Parallel & Perpendicular Lines

It's all in the slopes

Parallel: $m_1 = m_2$ **Perpendicular:** $m_1 \cdot m_2 = -1$ (negative reciprocals, for nonvertical lines)

Tutor's Note

Parallel lines never meet, so they must rise at the exact same rate — equal slopes. Perpendicular lines cross at a right angle, and it turns out their slopes are "negative reciprocals": flip the fraction and change the sign. So a slope of 2 is perpendicular to $-\frac{1}{2}$. A quick check: multiply them and you should get -1.

LINES

A vertical line and a horizontal line are perpendicular, but the vertical line has undefined slope, so the product rule does not apply to that pair.

Distance & Midpoint

Distance Formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Tutor's Note

The distance formula is secretly the Pythagorean theorem in disguise: the horizontal gap and vertical gap are the two legs of a right triangle, and the distance is the hypotenuse. The midpoint is even friendlier — it's just the average of the two x's and the average of the two y's, landing you exactly halfway.

Direct & Inverse Variation

Direct: $y = kx$

Inverse: $y = \dfrac{k}{x} \quad (x \neq 0,\ k \neq 0)$, or $xy = k$

Tutor's Note

Direct variation is a straight line through the origin — double the input, double the output when k is positive (think hours worked and pay earned). If k is negative, the line still passes through the origin but goes downward. **Inverse** variation is a constant-product trade-off — with positive quantities, more speed means less travel time. The constant k stays fixed and ties the two quantities together.

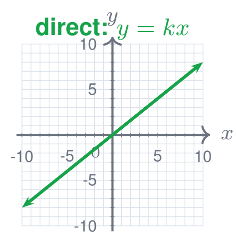
Visual: direct vs. inverse variation

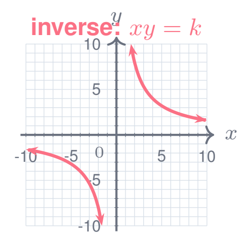
Compare the shapes

Direct variation is a line through the origin: $y = kx$.

Inverse variation bends in two branches because $xy = k$ stays fixed.

Both graphs continue beyond the viewing window.





4 Exponents & Scientific Notation

An exponent is just a shorthand for repeated multiplication. Once you see that, every exponent rule below stops being something to memorize and becomes something you can figure out.

The Laws of Exponents

These rules apply whenever every expression involved is defined. In this list, m and n are integers unless a fractional exponent is shown.

Product
$a^m \cdot a^n = a^{m+n}$

Power of a Power
$(a^m)^n = a^{mn}$

Quotient
$\dfrac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$

Power of a Product
$(ab)^n = a^n b^n$

Power of a Quotient
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad (b \neq 0)$

Negative Exponent
$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$

Zero Exponent
$a^0 = 1 \quad (a \neq 0)$

Fractional Exponent
$a^{m/n} = (\sqrt[n]{a})^m \quad (n \in \mathbb{Z}^+)$

Real-number note: with even roots, the base under the root must be nonnegative. For example, $\sqrt{x^2} = |x|$, not always x. With odd roots, negative radicands are allowed.

Tutor's Note

Why these all make sense $a^3 \cdot a^2$ means $(a \cdot a \cdot a)(a \cdot a)$ — five a's, so you *add* the exponents. Dividing cancels factors, so you *subtract*. A power of a power stacks groups, so you *multiply*. The strange ones follow the same logic: $a^0 = 1$ because $\frac{a^3}{a^3} = a^{3-3} = a^0$ and anything over itself is 1. A *negative* exponent just means "flip it to the other side of the fraction." Memorize the pattern once and the rest is bookkeeping.

Example

$\frac{2x^3 \cdot 3x^4}{x^2} = \frac{6x^7}{x^2} = 6x^5$. (Multiply coefficients, add then subtract exponents.)

Scientific Notation

A compact way to write very big or very small numbers without drowning in zeros.

Standard form

$$a \times 10^n, \qquad 1 \le |a| < 10, \quad n \in \mathbb{Z}$$

Big numbers: $n > 0$. Numbers with $0 < |x| < 1$: $n < 0$. Zero is written as 0.

Tutor's Note

The power of 10 just counts how many places the decimal point moves. Positive powers push it right (making the number bigger); negative powers pull it left (smaller). To multiply two of these, multiply the front numbers and *add* the exponents; to divide, divide the fronts and *subtract* — exactly the exponent laws from above.

Example

$3{,}400 = 3.4 \times 10^3 \qquad 0.0052 = 5.2 \times 10^{-3}$.
$(2 \times 10^5)(3 \times 10^2) = 6 \times 10^7$.

Tip: The front number always has exactly one digit before the decimal point — that's what makes it "standard."

5 Polynomials & Factoring

A polynomial is just a sum of terms like $3x^2$, $-5x$, and 7. Multiplying them out and then factoring them back apart are two sides of the same coin — and factoring is the key that unlocks quadratics in the next section.

Polynomial Vocabulary

Degree — the highest exponent in a nonzero polynomial.
Leading coefficient — the number on the highest-degree term.

Named by number of terms

- Monomial — 1 term
- Binomial — 2 terms
- Trinomial — 3 terms

Tutor's Note

"Like terms" are terms with the exact same variable part — $3x^2$ and $5x^2$ are like terms, but $3x^2$ and $3x$ are not. You can only combine like terms, the same way you can add apples to apples but not apples to oranges.

Adding, Subtracting & Multiplying

Add / Subtract

Combine like terms.

$$(3x^2 + 2x) - (x^2 - 5x) = 2x^2 + 7x$$

Multiply — FOIL for two binomials

$$(a + b)(c + d) = ac + ad + bc + bd$$

First, **O**uter, **I**nner, **L**ast.

Tutor's Note

When subtracting, distribute the minus sign to *every* term in the second parentheses — that sign flip is the #1 source of errors here. For multiplying, FOIL is really just the distributive property done twice: every term in the first group shakes hands with every term in the second.

SIGNS

$-(x^2 - 5x)$ becomes $-x^2 + 5x$. A minus sign outside parentheses distributes to every term inside.

Special Products (memorize these — they save time)

Square of a Sum
$(a+b)^2 = a^2 + 2ab + b^2$

Difference of Squares
$(a+b)(a-b) = a^2 - b^2$

Square of a Difference
$(a-b)^2 = a^2 - 2ab + b^2$

Sum/Diff of Cubes
$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

Tutor's Note

These are just FOIL results worth remembering. Notice the "difference of squares" magic: the middle terms cancel, leaving only $a^2 - b^2$. And watch the common slip — $(a+b)^2$ is *not* $a^2 + b^2$; you must include that $2ab$ middle term.

Factoring — a Reliable Game Plan

Factoring reverses multiplication: it rewrites a polynomial as a product. Always work through these steps in order.

The factoring checklist

1. **GCF first.** Pull out the greatest common factor from every term.
2. **Count the terms:**
   - 2 terms $\rightarrow$ try difference of squares or sum/difference of cubes
   - 3 terms $\rightarrow$ trinomial factoring
   - 4 terms $\rightarrow$ factor by grouping
3. **Check** by multiplying your answer back out.

Tutor's Note

Factoring feels like a puzzle at first, but the checklist removes the guesswork. Step one is non-negotiable: always look for a common factor before anything else — it makes every later step easier. Then let the number of terms point you to the right tool.

Factoring Trinomials

When $a = 1$: find two numbers that *multiply* to c and *add* to b.

$$x^2 + bx + c = (x+p)(x+q), \qquad p+q=b,\ pq=c$$

When $a \neq 1$: use the *ac-method* — split the middle term, then group.

Example

$x^2 + 7x + 12$: we need two numbers that multiply to 12 and add to 7 — that's 3 and 4. So $x^2 + 7x + 12 = (x+3)(x+4)$.

Tutor's Note

This "multiply to c, add to b" trick works because when you FOIL $(x+p)(x+q)$ you get $x^2+(p+q)x+pq$. So the outside number c is the product and the middle number b is the sum. List the factor pairs of c and find the pair that adds up right.

6 Quadratic Functions & Equations

A quadratic has an x^2 in it, and its graph is a graceful U-shaped curve called a parabola. This is the showpiece of Algebra 1 — and once you have factoring and the quadratic formula, you can solve any of them.

Standard & Vertex Forms

Standard Form
$y = ax^2 + bx + c \quad (a \neq 0)$

Vertex Form
$y = a(x-h)^2 + k$
vertex at (h, k)

Tutor's Note

Both forms describe the same parabola. **Standard form** shows the y-intercept at a glance (it's c). **Vertex form** hands you the most important point — the very top or bottom of the U — directly as (h, k). The number a controls the shape: large a makes a narrow U, small a a wide one, and a negative a flips it upside down.

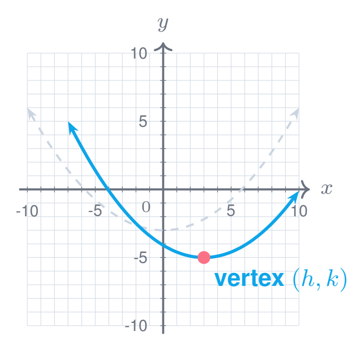
Visual: vertex form shifts the parabola

Vertex form

$y = a(x-h)^2 + k$ puts the vertex at (h, k).

The highlighted parabola has vertex $(3, -5)$, so the graph shifts right and down.

The arrows show each arm keeps rising outside the window.



Reading the Graph (the Parabola)

Key features

- Opens **up** if $a > 0$, **down** if $a < 0$.
- Axis of symmetry: $x = -\dfrac{b}{2a}$
- Vertex: plug that x back in to get y.
- y-intercept: $(0,\ c)$

Tutor's Note

Every parabola is perfectly symmetric, like a mirror folded down the middle. That mirror line is the axis of symmetry, found by $x = -\frac{b}{2a}$. The vertex sits right on that line — it's the lowest point if the U opens up, or the highest if it opens down. That makes the vertex the key to any "maximum height" or "minimum cost" problem.

Visual: key parabola features

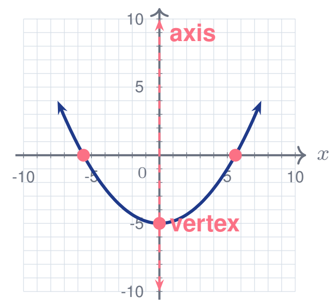
Parabola checklist

Find the axis first: $x = -\frac{b}{2a}$.

The vertex sits on the axis. The x-intercepts are where the curve crosses the x-axis.

Since $a > 0$, the parabola opens upward.

opens up if $a > 0$



Solving Quadratic Equations

The Quadratic Formula — it always works

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (a \neq 0)$$

Tutor's Note

You have a toolbox of methods — reach for the easiest one that fits:

- **Factoring** (fastest *when* it factors): if $pq = 0$, then $p = 0$ or $q = 0$.
- **Square roots:** if $x^2 = k$, then $x = \pm\sqrt{k}$ for $k \geq 0$.
- **Completing the square:** rewrites it in vertex form.

Completing the square

$$x^2 + bx = \left(x + \tfrac{b}{2}\right)^2 - \left(\tfrac{b}{2}\right)^2$$

Tutor's Note

The "zero-product" idea is sneaky-powerful: if two things multiply to zero, at least one of them must *be* zero. That's why factoring solves equations — set each factor to zero. When factoring fails, the quadratic formula never does; it's the universal key that works on every quadratic, no matter how ugly.

♦ The Discriminant — a Sneak Peek at the Answers

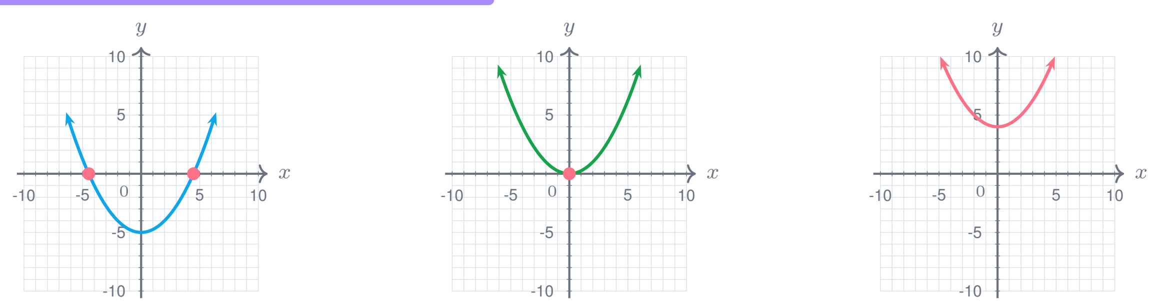
$D = b^2 - 4ac$ counts the real solutions

- $D > 0$: two different real solutions
- $D = 0$: one repeated real solution
- $D < 0$: no real solutions (the parabola misses the x-axis)

Tutor's Note

The discriminant is the part under the square root. Before solving, it tells you what to expect: a positive value means the parabola crosses the x-axis twice, zero means it just touches once, and a negative means it floats entirely above or below — no real crossing, because you can't take the square root of a negative in real numbers.

Visual: discriminant and x-intercepts



$D > 0$: **two x-intercepts** — $D = 0$: **one touch** — $D < 0$: **no real intercepts**

Example

$x^2 - 5x + 6 = 0$. It factors: $(x-2)(x-3) = 0$, so $x = 2$ or $x = 3$. Check: $D = (-5)^2 - 4(1)(6) = 25 - 24 = 1 > 0$, confirming two real answers.

7 Radicals & Rational Expressions

A radical (square root) undoes a square, and a rational expression is just a fraction with variables. Both follow rules you already know from numbers — we're simply letting letters join in.

Radical Rules

Product
$\sqrt{ab} = \sqrt{a}\,\sqrt{b} \quad (a, b \geq 0)$

Radical = Exponent
$\sqrt[n]{a} = a^{1/n} \quad (n \in \mathbb{Z}^+)$

Quotient
$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (a \geq 0,\ b > 0)$

Power
$a^{m/n} = (\sqrt[n]{a})^m$

Real-number note: even roots need a nonnegative radicand. Also, $\sqrt{x^2} = |x|$ because the principal square root is never negative. Odd roots can have negative radicands.

Tutor's Note

The most freeing idea here: a root is just a *fractional exponent*. A square root is the $\frac{1}{2}$ power, a cube root is the $\frac{1}{3}$ power. Once you see that, all the exponent laws from Section 4 apply to radicals too — you don't need a separate rule book. The product rule also lets you "break apart" a root to simplify it.

Simplifying & Rationalizing

Simplify a radical

Pull out perfect-square factors: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

Rationalize the denominator

Multiply by a clever form of 1 to clear the root from the bottom:

$$\frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a} \quad (a > 0) \qquad \frac{1}{a+\sqrt{b}} \cdot \frac{a-\sqrt{b}}{a-\sqrt{b}} = \frac{a-\sqrt{b}}{a^2-b} \quad (b \geq 0,\ a^2 \neq b)$$

The flipped-sign partner $a - \sqrt{b}$ is called the **conjugate**.

Tutor's Note

To simplify, hunt for the biggest perfect square hiding inside (4, 9, 16, 25, . . .) and take its root out front. "Rationalizing" just means cleaning roots out of the denominator, which is considered tidy form. Multiplying by the conjugate works because of difference of squares — it turns $\sqrt{b}$ into plain b and the messy root vanishes.

Solving Radical Equations

Isolate, then undo the root

Get the radical alone, raise both sides to the matching power, then **always check** your answers in the original equation.

$$\sqrt{x+3} = 5 \Rightarrow x + 3 = 25 \Rightarrow x = 22$$

Tutor's Note

Squaring both sides is the move that frees x from under the root. But squaring can secretly introduce a fake answer (an "extraneous" solution), so checking isn't optional here — it's part of the method. Plug your answer back in; if it doesn't actually work, toss it.

CHECK

After squaring both sides, check in the *original* equation, not the squared version. The squared equation can accidentally accept fake answers.

Rational Expressions

Simplify by canceling common factors:

$$\frac{x^2-9}{x+3} = \frac{(x-3)(x+3)}{x+3} = x - 3, \qquad x \neq -3$$

Multiply / **Divide** like any fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \quad (b, d, c \neq 0)$$

Restriction: a denominator can never equal 0. Exclude any x that would make the bottom zero — those values are off-limits.

Tutor's Note

To simplify a rational expression, factor the top and bottom first, then cancel matching factors — exactly like reducing $\frac{6}{9}$ to $\frac{2}{3}$. Just remember the one rule the calculator can't bend: dividing by zero is undefined, so always note which x-values are forbidden before you simplify.

8 Systems of Equations & Inequalities

A system is two (or more) equations that must be true at the same time. The solution is the point where they agree — where their graphs cross.

Three Ways to Solve

Graphing

Plot both lines; the solution is the point where they intersect.

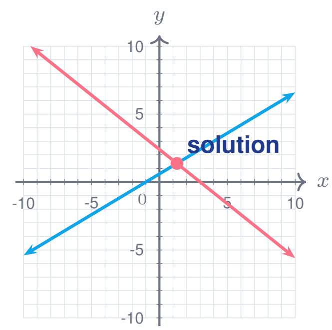
Visual: solution is the intersection

Graphing a system

Draw both equations on the same plane.

The solution is the point that lies on both lines at the same time.

If the lines do not meet, there is no solution. If they are the same line, there are infinitely many.



Substitution

Solve one equation for a variable, then plug it into the other.

$$\begin{cases} y = 2x \\ x + y = 6 \end{cases} \Rightarrow x + 2x = 6 \Rightarrow x = 2,\ y = 4$$

Elimination

Add or subtract the equations to cancel a variable.

$$\begin{cases} x + y = 7 \\ x - y = 1 \end{cases} \Rightarrow 2x = 8 \Rightarrow x = 4,\ y = 3$$

Tutor's Note

Which method should I pick? **Graphing** is great for seeing what's going on, but it's only as exact as your drawing. **Substitution** shines when one variable is already alone (like $y = 2x$). **Elimination** is fastest when the equations line up so a variable cancels when you add them. All three reach the same answer — choose whichever fits the problem in front of you.

How Many Solutions?

One solution
Lines cross once.
Different slopes.

Infinitely many
Same line.
Same slope and same intercept.

No solution
Parallel lines.
Same slope, different intercept.

Tutor's Note

Think geometrically. Two lines that lean differently must cross at exactly one point — one solution. Two parallel lines never meet — no solution. And if both equations secretly describe the *same* line, every point works — infinitely many. The slopes and intercepts tell you which case you're in before you even solve.

Systems of Inequalities

Graph, then shade

- Solid line for $\leq$, $\geq$; dashed line for $<$, $>$.
- Shade the side of each line that makes its inequality true.
- The **solution** is the region where the shadings overlap.

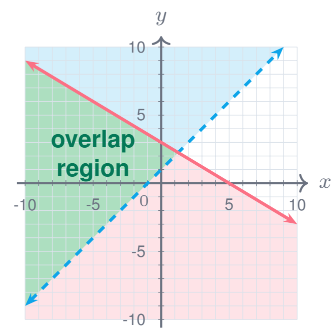
Visual: overlapping solution region

Shade and overlap

Use a dashed boundary for $<$ or $>$ and a solid boundary for $\leq$ or $\geq$.

The answer is the region where both shaded half-planes overlap.

Test a point, often $(0, 0)$, to decide which side to shade.



Tutor's Note

With inequalities, the answer is a whole *region*, not one point. A solid boundary counts; a dashed boundary does not. The overlapping shaded zone is the solution. **Tip:** To choose a side, test $(0, 0)$ if it is not on the boundary.

9 Sequences & Statistics

We finish with two practical topics: patterns of numbers (sequences) and ways to summarize data (statistics). Both show up constantly in real life and on tests.

Arithmetic Sequences

Add the same amount each time (d)

$$a_n = a_1 + (n-1)d$$

a_1 is the first term and d is the common difference.

Tutor's Note

An arithmetic sequence grows by *adding* a fixed step every time — like stairs of equal height. The formula just says: start at a_1, then take $(n-1)$ steps of size d to reach the nth term. The $(n-1)$ catches a lot of students — the first term takes *zero* steps, so we subtract one.

Example

$3, 7, 11, 15, \ldots$ here $a_1 = 3$ and $d = 4$, so $a_n = 3 + (n-1)4 = 4n - 1$.

Geometric Sequences

Multiply by the same amount each time (r)

$$a_n = a_1 \cdot r^{n-1}$$

r is the common ratio.

Tutor's Note

A geometric sequence grows by *multiplying* by a fixed ratio — this is how money compounds and how populations grow. Same structure as before, but repeated multiplication replaces repeated addition, so r is raised to the $(n-1)$ power.

Example

$2, 6, 18, 54, \ldots$ here $a_1 = 2$ and $r = 3$, so $a_n = 2 \cdot 3^{n-1}$.

Measures of Center & Spread

Mean (average)

$$\bar{x} = \frac{\sum x}{n}$$

Median

the middle of ordered data; if there are two middles, average them

Mode

the most frequent value

Range

$\max - \min$

1 Functions, Transformations & Inverses

Core function formulas

Function notation	$y = f(x)$ means the output when the input is x.
Average rate of change	$\dfrac{f(b) - f(a)}{b - a}$, where $b \neq a$.
Composition	$(f \circ g)(x) = f(g(x))$. Use the domain of g and any added restrictions from f.
Inverse functions	$f^{-1}(x)$ reverses $f(x)$, and $f(f^{-1}(x)) = x$ on the allowed domain.
Transformation model	$g(x) = a\,f(b(x - h)) + k$. Vertical shift k, horizontal shift h, vertical scale $\lvert a\rvert$, horizontal scale $\frac{1}{\lvert b\rvert}$.

Tutor's Note

Algebra 2 functions are machines you can move, stretch, combine, and reverse. Always track domain: denominators cannot be zero, even roots need nonnegative radicands, and logs need positive arguments.

Example

If $f(x) = x^2$ and $g(x) = 2f(x-3) - 5$, then $g(x) = 2(x-3)^2 - 5$. The graph shifts right 3, stretches vertically by 2, and shifts down 5.

INVERSE

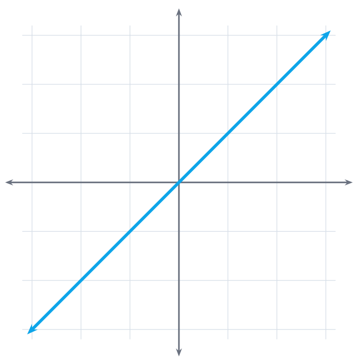
To find an inverse, write $y = f(x)$, swap x and y, solve for y, and then write $f^{-1}(x)$. If the original function is not one-to-one, restrict the domain before taking an inverse.

Before you move on A function rule, a line, and a system all answer the same question in different ways: **what output matches this input?** Check the input restrictions first, then use slope, substitution, or graph intersections to compare outputs.

2 Parent Functions & Transformations

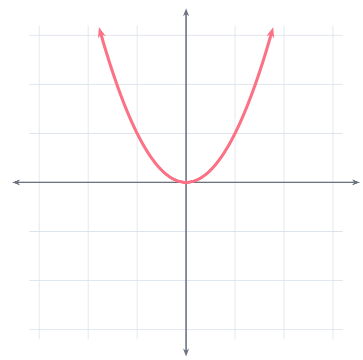
Every Algebra 2 graph is a transformed version of one of these parent shapes. Learn the parents — their graphs, domains, and ranges — then apply shifts, reflections, and stretches. Each curve continues past the arrows.

Linear $y = x$



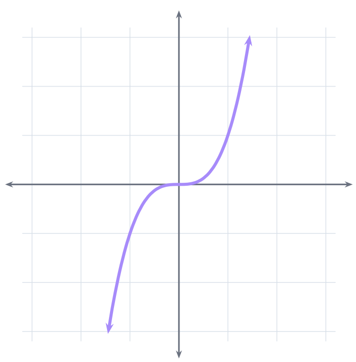
Domain $\mathbb{R}$ • Range $\mathbb{R}$

Quadratic $y = x^2$



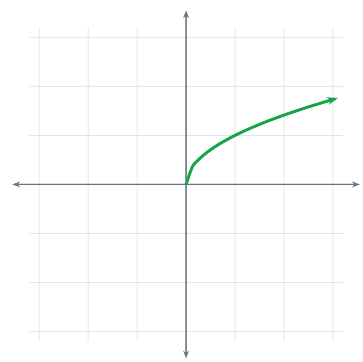
Domain $\mathbb{R}$ • Range $y \geq 0$

Cubic $y = x^3$



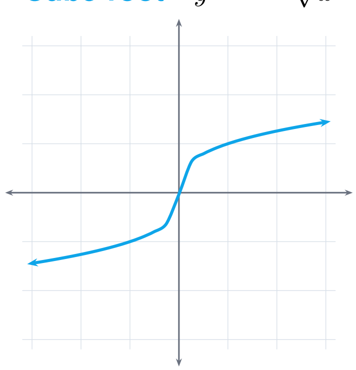
Domain $\mathbb{R}$ • Range $\mathbb{R}$

Square root $y = \sqrt{x}$



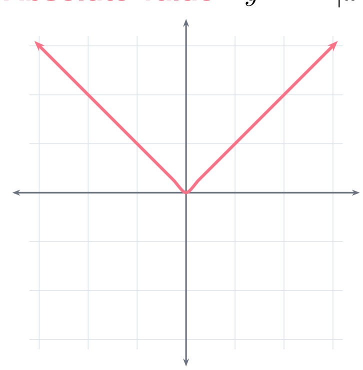
Domain $x \geq 0$ • Range $y \geq 0$

Cube root $y = \sqrt[3]{x}$



Domain $\mathbb{R}$ • Range $\mathbb{R}$

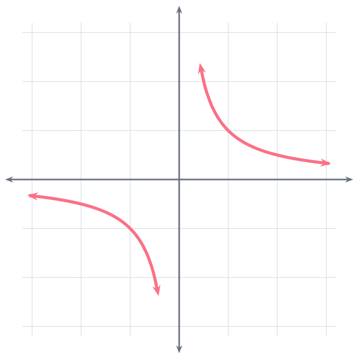
Absolute value $y = |x|$



Domain $\mathbb{R}$ • Range $y \geq 0$

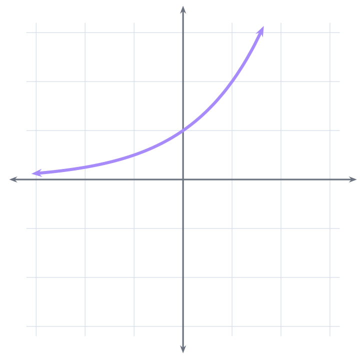
More parent functions

Reciprocal $y = \frac{1}{x}$



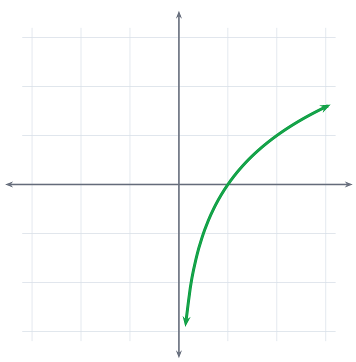
Domain $x \neq 0$ • Range $y \neq 0$

Exponential $y = b^x$



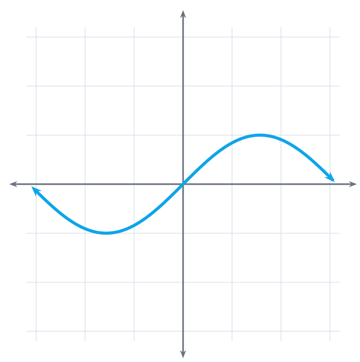
Domain $\mathbb{R}$ • Range $y > 0$

Logarithm $y = \log_b x$



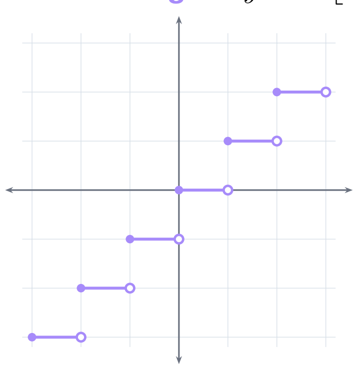
Domain $x > 0$ • Range $\mathbb{R}$

Sine $y = \sin x$



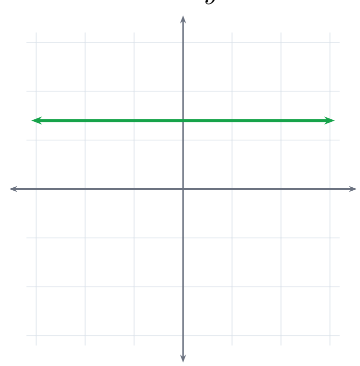
Domain $\mathbb{R}$ • Range $[-1, 1]$

Greatest integer $y = \lfloor x \rfloor$



Domain $\mathbb{R}$ • Range integers

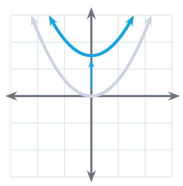
Constant $y = b$



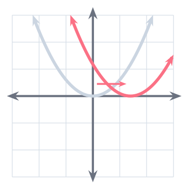
Domain $\mathbb{R}$ • Range $\{b\}$

Transformations of $y = x^2$

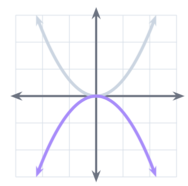
The gray curve is the parent $y = x^2$. Each colored curve shows one change from $g(x) = a\,f(x-h) + k$.



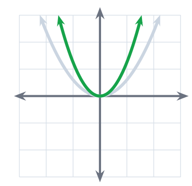
$+k$: up 1.5



$x - h$: right 1.4



$a < 0$: reflect



$a > 1$: stretch

Transformation rules for $g(x) = a\,f\big(b(x-h)\big) + k$

Vertical shift	$+k$ moves the graph up k; $-k$ moves it down.
Horizontal shift	$(x-h)$ moves the graph right h; $(x+h)$ moves it left.
Vertical stretch/shrink	$\lvert a\rvert > 1$ stretches; $0 < \lvert a\rvert < 1$ shrinks toward the x-axis.
Horizontal stretch/shrink	factor $\frac{1}{\lvert b\rvert}$; $\lvert b\rvert > 1$ shrinks, $0 < \lvert b\rvert < 1$ stretches.
Reflections	$a < 0$ reflects over the x-axis; $b < 0$ reflects over the y-axis.
Order of operations	stretch/reflect first, then shift; horizontal changes work in reverse.

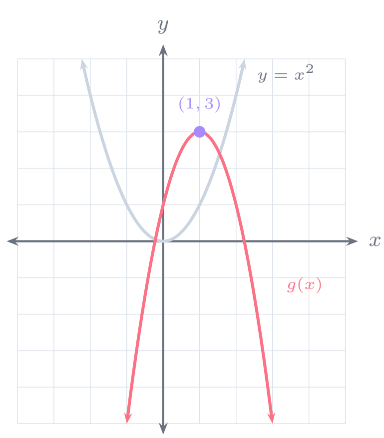
Tutor's Note

Read a transformed rule from the inside out. Inside the function, changes affect x and behave "backwards" (right/left, horizontal scaling). Outside the function, changes affect y and behave as written (up/down, vertical scaling, reflection).

DOMAIN & RANGE

Transformations move domain and range too. A vertical shift changes the range; a horizontal shift changes the domain; reflections can swap which way an inequality points.

Worked example: build $g(x) = -2(x-1)^2 + 3$ from the parent $y = x^2$



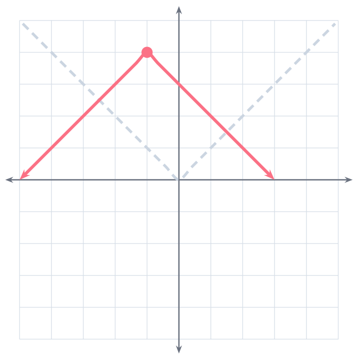
Read $g(x) = -2(x-1)^2 + 3$ as four moves on the parent $y = x^2$:

1. $x - 1$ inside: shift **right** 1.
2. $\times 2$: **stretch** vertically (narrower).
3. leading $-$: **reflect** over the x-axis (opens down).
4. $+3$ outside: shift **up** 3, so the vertex lands at $(1, 3)$.

Transforming every parent function

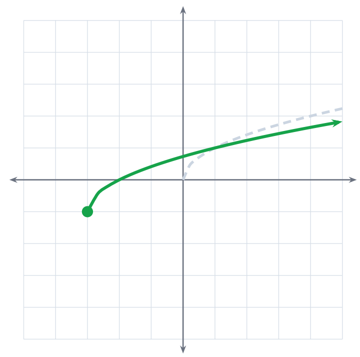
The gray dashed curve is always the parent; the **colored** curve is the transformed function with its key point marked. The same moves — shift, reflect, stretch — work on every family, so once you can read one, you can read them all.

Absolute value $y = |x| \to -|x+1| + 4$



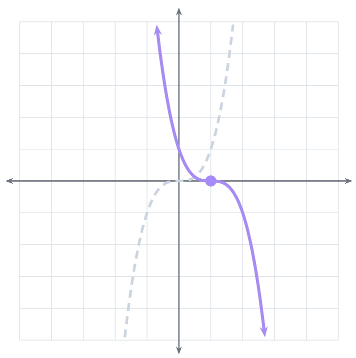
Reflect down, left 1, up 4 — vertex $(-1, 4)$.

Square root $y = \sqrt{x} \to \sqrt{x+3} - 1$



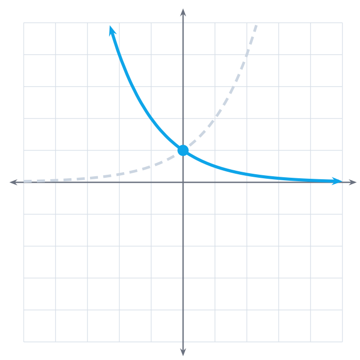
Left 3, down 1 — starts at $(-3, -1)$.

Cubic $y = x^3 \to -(x-1)^3$



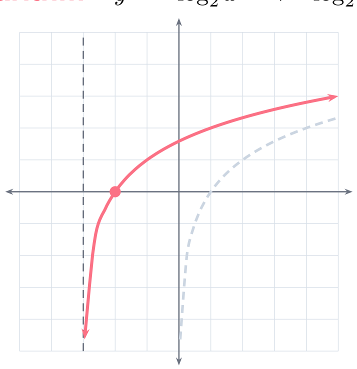
Reflect, right 1 — inflection at $(1, 0)$.

Exponential $y = 2^x \to 2^{-x}$



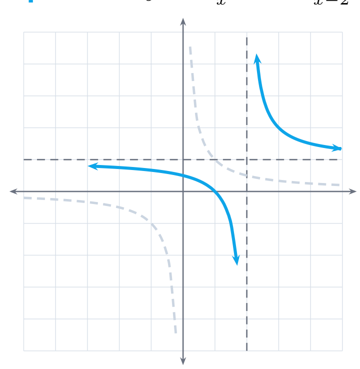
Reflect over the y-axis — same asymptote $y = 0$.

Logarithm $y = \log_2 x \to \log_2(x+3)$



Left 3 — asymptote moves to $x = -3$.

Reciprocal $y = \frac{1}{x} \to \frac{1}{x-2} + 1$



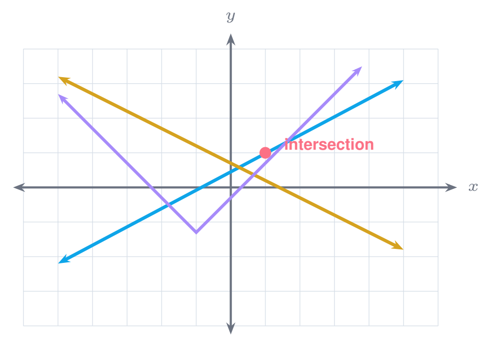
Right 2, up 1 — asymptotes $x = 2$, $y = 1$.

3 Linear, Absolute Value & Systems

Quick visual: line, V-shape, or intersection?

Lines track constant rate, absolute value graphs measure distance from a center, and systems ask where two rules agree. Before solving, sketch the story in one sentence.

Slope $m = \frac{\text{change in } y}{\text{change in } x}$ **System** shared point



Lines and systems

Slope	$m = \frac{y_2 - y_1}{x_2 - x_1}$, with $x_2 \neq x_1$.
Slope-intercept form	$y = mx + b$.
Point-slope form	$y - y_1 = m(x - x_1)$.
Standard form	$Ax + By = C$, usually with integer coefficients.
Parallel lines	Same slope.
Perpendicular lines	Slopes multiply to -1, unless one line is vertical and the other is horizontal.
Absolute value equation	$\lvert x - a\rvert = b$ gives $x = a \pm b$ when $b \geq 0$. If $b < 0$, no solution.
System types	One intersection: one solution. Parallel distinct lines: no solution. Same line: infinitely many solutions.

Tutor's Note

Systems ask for values that make every equation true at the same time. Substitution is great when one variable is already isolated. Elimination is great when coefficients can be made opposites.

Example

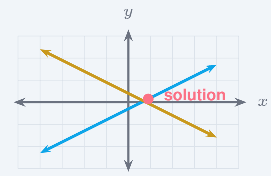
Solve $\begin{cases} 2x + y = 7 \\ x - y = 2 \end{cases}$. Add the equations to get $3x = 9$, so $x = 3$. Then $3 - y = 2$, so $y = 1$. The solution is $(3, 1)$.

INEQUALITIES

When multiplying or dividing an inequality by a negative number, reverse the inequality symbol. For $\lvert x - a\rvert < b$, write $a - b < x < a + b$. For $\lvert x - a\rvert > b$, write $x < a - b$ or $x > a + b$.

System story check

Before solving, predict the answer count: crossing lines mean **one** solution, parallel lines mean **none**, and the same line means **infinitely many**. Choose substitution when a variable is isolated; choose elimination when coefficients can cancel.



Intersection = shared answer

System solving choice Is one variable already alone? Use substitution. Do matching coefficients appear? Use elimination. Are the equations in graph form? Compare slopes and intercepts first.

4 Quadratics & Complex Numbers

Quadratic toolbox

Standard form	$f(x) = ax^2 + bx + c$, where $a \neq 0$.
Vertex & axis	$h = -\dfrac{b}{2a}$, $k = f(h)$; axis of symmetry $x = h$.
Vertex form	$f(x) = a(x-h)^2 + k$. Opens up if $a > 0$ (min), down if $a < 0$ (max).
Factored form	$f(x) = a(x - r_1)(x - r_2)$ shows zeros r_1, r_2.
Quadratic formula	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
Discriminant	$D = b^2 - 4ac$: $D > 0$ two real roots, $D = 0$ one real double root, $D < 0$ two complex roots.
Complex unit	$i^2 = -1$, so $\sqrt{-a} = i\sqrt{a}$ for $a > 0$.
Powers of i	cycle every 4: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, then repeat.
Complex conjugate	The conjugate of $a + bi$ is $a - bi$; $(a+bi)(a-bi) = a^2 + b^2$.

Tutor's Note

The same quadratic can be useful in different forms. Standard form is good for the quadratic formula and the y-intercept. Vertex form is best for maximum/minimum. Factored form is best for zeros.

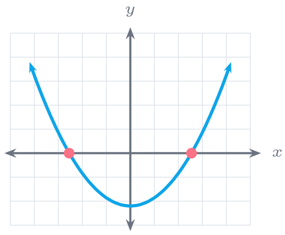
Example

For $f(x) = 2x^2 - 8x + 3$, the vertex has $h = -\frac{-8}{2(2)} = 2$ and $k = f(2) = 8 - 16 + 3 = -5$. Vertex: $(2, -5)$.

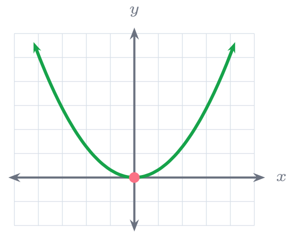
COMPLEX

When $D < 0$, the graph has no real x-intercepts, but the equation still has complex solutions. Keep $i^2 = -1$ handy when simplifying.

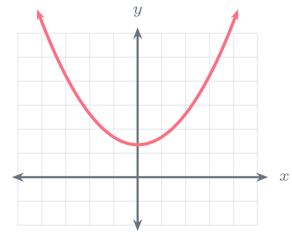
Discriminant snapshot Let $D = b^2 - 4ac$. It tells how many real x-intercepts the parabola has before you solve.



$D > 0$: **two**



$D = 0$: **one**



$D < 0$: **none real**

5 Polynomial Functions & Factoring

Polynomial formulas

Degree	Highest exponent after the polynomial is simplified.
Square of a binomial	$(a \pm b)^2 = a^2 \pm 2ab + b^2$.
Cube of a binomial	$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
Difference of squares	$a^2 - b^2 = (a+b)(a-b)$.
Sum/difference of cubes	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$; $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
Division algorithm	$P(x) = D(x)\,Q(x) + R(x)$, with $\deg R < \deg D$ and $D(x) \neq 0$.
Remainder theorem	When $P(x)$ is divided by $x - c$, the remainder is $P(c)$. For $ax - b$, it is $P\left(\frac{b}{a}\right)$.
Factor theorem	$x - c$ is a factor of $P(x)$ exactly when $P(c) = 0$.
Rational root candidates	For integer coefficients, candidates are $\pm\frac{p}{q}$ where p divides the constant and q divides the leading coefficient.
Fundamental theorem	A degree-n polynomial has exactly n complex roots, counting multiplicity.
Multiplicity	Even multiplicity: graph is tangent to (bounces off) the x-axis. Odd multiplicity: graph crosses, flattening if > 1.
End behavior	The leading term $a_n x^n$ controls the far left and far right ends.

Tutor's Note

Factoring is reverse multiplication. First take out a GCF, then look for special patterns, grouping, or trinomial factoring. For higher-degree polynomials, test possible rational zeros and use division to reduce the degree.

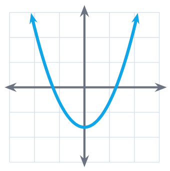
Example

Factor $x^3 - 8$. This is a difference of cubes: $x^3 - 2^3 = (x-2)(x^2 + 2x + 4)$.

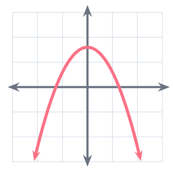
END BEHAVIOR

Even degree polynomials point the same way on both ends. Odd degree polynomials point opposite ways. A positive leading coefficient makes the right end go up; a negative leading coefficient makes the right end go down.

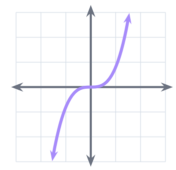
End behavior: degree & leading coefficient Only the leading term $a_n x^n$ controls the far ends. Even degree: both ends match. Odd degree: ends go opposite ways. The sign of a_n sets the right end.



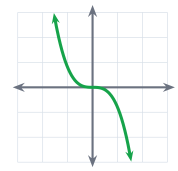
Even, $a_n > 0$



Even, $a_n < 0$



Odd, $a_n > 0$



Odd, $a_n < 0$

6 Rational Expressions & Variation

Rational expression rules

Restriction	Any value that makes a denominator 0 is not allowed.
Multiply	$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, with $b, d \neq 0$.
Divide	$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$, with $b, c, d \neq 0$.
Add/subtract	$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, with $b, d \neq 0$.
Direct variation	$y = kx$.
Inverse variation	$y = \frac{k}{x}$ or $xy = k$, with $x \neq 0$.
Joint variation	$y = kxz$.
Combined variation	$y = \frac{kx}{z}$, with $z \neq 0$.

Tutor's Note

Simplify rational expressions by factoring first, then cancel common factors. Never cancel pieces across addition or subtraction. For rational equations, multiply by the LCD, solve, then check every answer in the original equation.

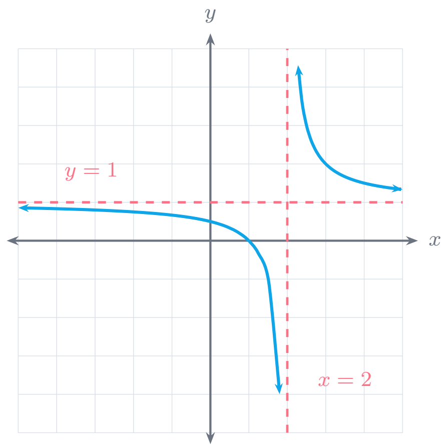
Example

$$\frac{x^2 - 9}{x^2 - 3x} = \frac{(x-3)(x+3)}{x(x-3)} = \frac{x+3}{x},$$ but the original restrictions are $x \neq 0$ and $x \neq 3$.

ASYMPTOTES

Vertical asymptotes often come from denominator zeros that remain after simplifying. Horizontal asymptotes compare degrees: lower numerator degree gives $y = 0$; equal degrees give the ratio of leading coefficients.

Reading a rational graph Denominator zeros that survive give *vertical asymptotes*; comparing degrees gives the *horizontal asymptote*; a canceled factor leaves a *hole*.



Vertical asymptote: set the surviving denominator to 0 ($x = 2$ here).

Horizontal asymptote: compare degrees — bottom-heavy gives $y = 0$, equal degrees give the ratio of leading coefficients, top-heavy gives none (slant instead).

Hole: a factor that cancels from top and bottom removes a single point, not a whole line.

7 Radicals & Rational Exponents

Laws of exponents

Product of powers $a^m \cdot a^n = a^{m+n}$.

Quotient of powers $\frac{a^m}{a^n} = a^{m-n}$, with $a \neq 0$.

Power of a power $(a^m)^n = a^{mn}$.

Power of a product/quotient $(ab)^n = a^n b^n$ and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, with $b \neq 0$.

Zero exponent $a^0 = 1$, with $a \neq 0$.

Negative exponent $a^{-n} = \frac{1}{a^n}$ and $\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$, with nonzero bases.

Radical and exponent forms

Rational exponent $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$ when the real expression is defined.

Product rule $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ when both sides are real.

Quotient rule $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$, with $b \neq 0$.

nth root of a^n $\sqrt[n]{a^n} = a$ if n is odd; $\sqrt[n]{a^n} = |a|$ if n is even.

Even roots $\sqrt{x}$ means the principal, nonnegative square root. For real even roots, the radicand must be nonnegative.

Conjugates $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$.

Radical equations Isolate the radical, raise both sides to the index power, solve, then check.

Tutor's Note

Radicals and rational exponents are two languages for the same idea. Use exponent rules when powers are easier; use radical notation when the root structure is clearer.

Example

$\frac{5}{\sqrt{3}}$ is rationalized by multiplying by $\frac{\sqrt{3}}{\sqrt{3}}$: $\frac{5\sqrt{3}}{3}$.

CHECK

Squaring both sides can create extraneous solutions. Always plug radical equation answers into the original equation.

8 Exponential & Logarithmic Functions

Growth, decay, and logs

Exponential model	$y = ab^x$, where a is the starting value and b is the multiplier.
Growth/decay	$A = P(1+r)^t$ for growth; $A = P(1-r)^t$ for decay.
Compound interest	$A = P\left(1+\frac{r}{n}\right)^{nt}$.
Continuous growth	$A = Pe^{rt}$; population $N(t) = N_0e^{rt}$.
Log definition	$\log_b x = y \iff b^y = x$, where $b > 0$, $b \neq 1$, and $x > 0$.
Common & natural log	$\log x = \log_{10} x$ and $\ln x = \log_e x$.
Inverse properties	$\log_b 1 = 0$, $\log_b b = 1$, $b^{\log_b x} = x$, $\log_b(b^x) = x$.
Product rule	$\log_b(MN) = \log_b M + \log_b N$, where $M, N > 0$.
Quotient rule	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$, where $M, N > 0$.
Power rule	$\log_b(M^p) = p\log_b M$, where $M > 0$.
Change of base	$\log_b M = \frac{\log_a M}{\log_a b} = \frac{\ln M}{\ln b}$, where $a, b > 0$ and $a, b \neq 1$.

Tutor's Note

Logs undo exponentials. If the variable is in the exponent, use logs. If the variable is inside a log, rewrite or condense logs until the equation can be solved.

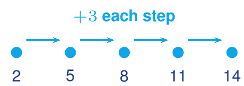
Example

Solve $3(2^x) = 24$. Divide by 3: $2^x = 8$. Since $8 = 2^3$, $x = 3$.

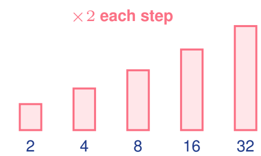
LOG DOMAIN

Every logarithm needs a positive argument. Check the original log equation, not just the simplified one.

Arithmetic vs. geometric at a glance Arithmetic sequences *add* the same step d; geometric sequences *multiply* by the same ratio r. A series is the sum of those terms.



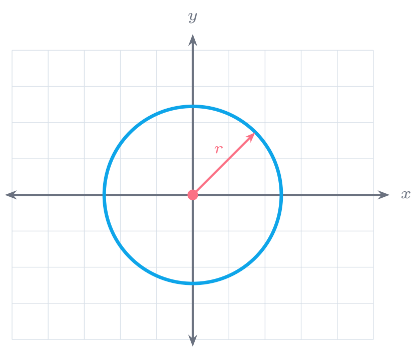
Arithmetic $a_n = a_1 + (n-1)d$



Geometric $a_n = a_1 r^{n-1}$

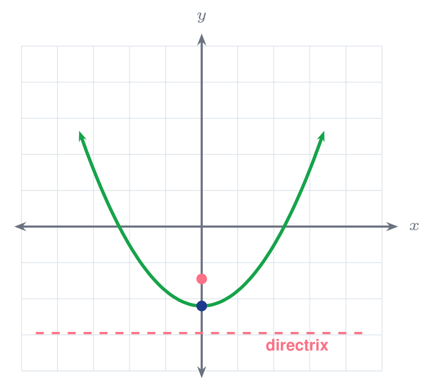
10 Conic Sections & Coordinate Formulas

Conic visual atlas Read a conic by asking: how many squared variables, plus or minus, and where are the center, vertex, foci, or asymptotes?



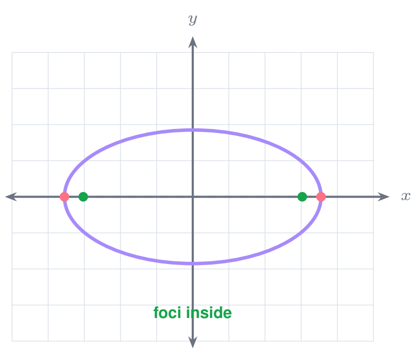
circle: center + radius

$$(x-h)^2 + (y-k)^2 = r^2$$



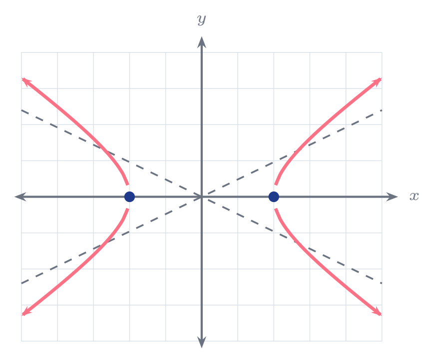
parabola: vertex + focus

$$(x-h)^2 = 4p(y-k)$$



ellipse: plus, closed oval

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

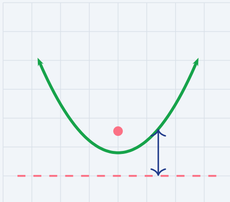


hyperbola: minus + asymptotes

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

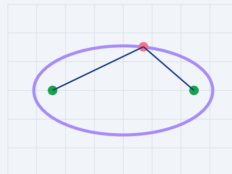
Why the shapes happen

Conics come from distance rules. The equation is easier to remember when the distance story is visible.



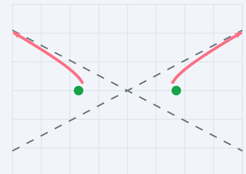
parabola

same distance to focus and directrix



ellipse

sum of distances to foci is constant



hyperbola

difference of distances to foci is constant

Coordinate and conic formulas

Distance	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
Midpoint	$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
Circle	$(x-h)^2 + (y-k)^2 = r^2$. Center (h,k), radius r.
Vertical parabola	$(x-h)^2 = 4p(y-k)$. Focus $(h, k+p)$, directrix $y = k - p$.
Horizontal parabola	$(y-k)^2 = 4p(x-h)$. Focus $(h+p, k)$, directrix $x = h - p$.
Ellipse	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The larger denominator gives the major axis.
Ellipse foci	If $a \geq b$, $c^2 = a^2 - b^2$ and foci are $(h \pm c, k)$. If $b > a$, foci are $(h, k \pm c)$.
Horizontal hyperbola	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. Vertices $(h \pm a, k)$, foci $(h \pm c, k)$.
Vertical hyperbola	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. Vertices $(h, k \pm a)$, foci $(h, k \pm c)$.
Hyperbola facts	$c^2 = a^2 + b^2$. Horizontal asymptotes: $y - k = \pm\frac{b}{a}(x-h)$. Vertical asymptotes: $y - k = \pm\frac{a}{b}(x-h)$.

CIRCLE
Both squared terms have equal coefficients; one radius.

PARABOLA
Only one variable is squared; p controls focus direction.

ELLIPSE
Two squared terms are added; foci stay inside the oval.

HYPERBOLA
Squared terms are subtracted; branches chase asymptotes.

Tutor's Note

Conics are graphs made from distances. The center or vertex tells you where the graph lives; the denominators and the value of p tell you how it opens or stretches.

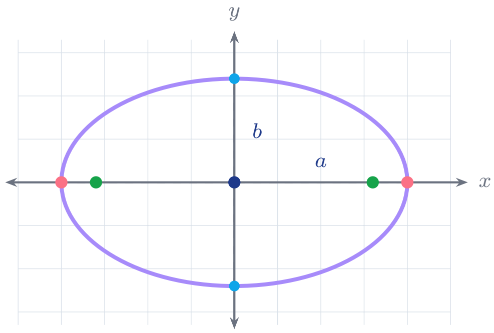
Example

$(x-2)^2 + (y+3)^2 = 25$ is a circle with center $(2, -3)$ and radius 5.

CONICS

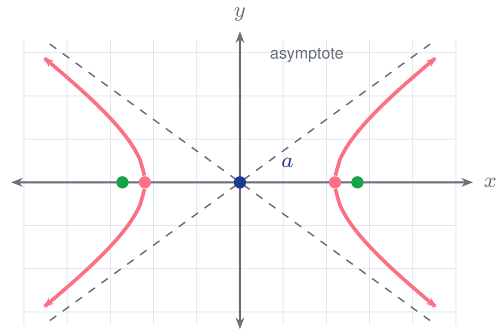
For ellipses, $c^2 = a^2 - b^2$. For hyperbolas, $c^2 = a^2 + b^2$. That sign change is one of the easiest places to lose accuracy.

Anatomy of an ellipse & a hyperbola The same parts appear in both: a center, two vertices, and two foci on the main axis. Only the sign and the focus rule change.



• center • vertex • focus • co-vertex

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \; c^2 = a^2 - b^2$

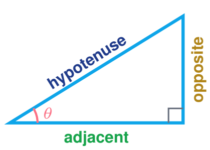


• center • vertex • focus

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \; c^2 = a^2 + b^2$

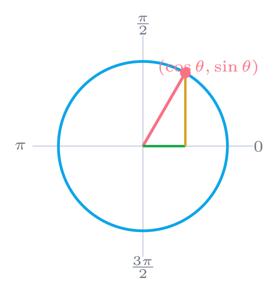
11 Trigonometry Essentials

Trigonometry visual atlas Trig connects a right triangle, the unit circle, radian measure, and repeating wave graphs. These are the same ratios seen four ways.

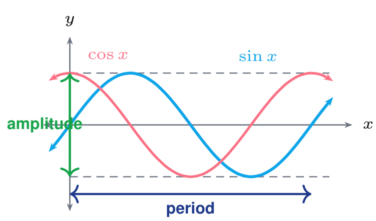


SOH-CAH-TOA

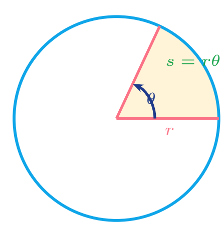
$\sin\theta = \frac{o}{h}, \quad \cos\theta = \frac{a}{h}, \quad \tan\theta = \frac{o}{a}$



unit circle: $x = \cos\theta$, $y = \sin\theta$



$y = A\sin(B(x-C)) + D$: amplitude $|A|$, period $\frac{2\pi}{|B|}$



radians measure arc per radius

$A_{\text{sector}} = \frac{1}{2}r^2\theta, \quad 180^\circ = \pi$ radians

Trig formulas

Coterminal angles	$\theta + 2\pi k$ radians or $\theta + 360^\circ k$, where k is any integer.
Radians/degrees	$\theta_{\text{rad}} = \theta_{\deg}\frac{\pi}{180}$ and $\theta_{\deg} = \theta_{\text{rad}}\frac{180}{\pi}$.
Right triangle ratios	$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.
Unit circle	$x = \cos\theta$, $y = \sin\theta$, and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ when $\cos\theta \neq 0$.
Pythagorean identity	$\sin^2\theta + \cos^2\theta = 1$.
Reciprocal identities	$\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\cot\theta = \frac{1}{\tan\theta}$.
Quotient identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$ when defined.
Trig graph	$y = A\sin(B(x - C)) + D$ has amplitude $\lvert A\rvert$, period $\frac{2\pi}{\lvert B\rvert}$, phase shift C, midline $y = D$.
Frequency	frequency $= \frac{1}{\text{period}} = \frac{\lvert B\rvert}{2\pi}$ — the number of cycles per unit.
Arc length and sector	$s = r\theta$ and $A = \frac{1}{2}r^2\theta$ when θ is in radians.
Law of sines	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
Law of cosines	$c^2 = a^2 + b^2 - 2ab\cos C$.
Triangle area	$K = \frac{1}{2}ab\sin C$ when sides a and b include angle C.

Special-angle values

Memorize quadrant I, then use signs by quadrant.

Angle	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Tutor's Note

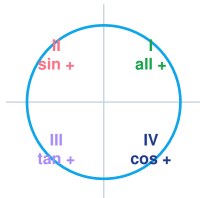
Trig connects angles, circles, and waves. In right triangles, use SOH-CAH-TOA. On the unit circle, cosine is the x-coordinate and sine is the y-coordinate.

Example

If $\theta = \frac{\pi}{3}$, then $\sin\theta = \frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$.

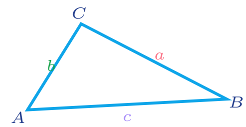
RADIANS

Arc length, sector area, and trig graph periods use radians unless the problem clearly says degrees.



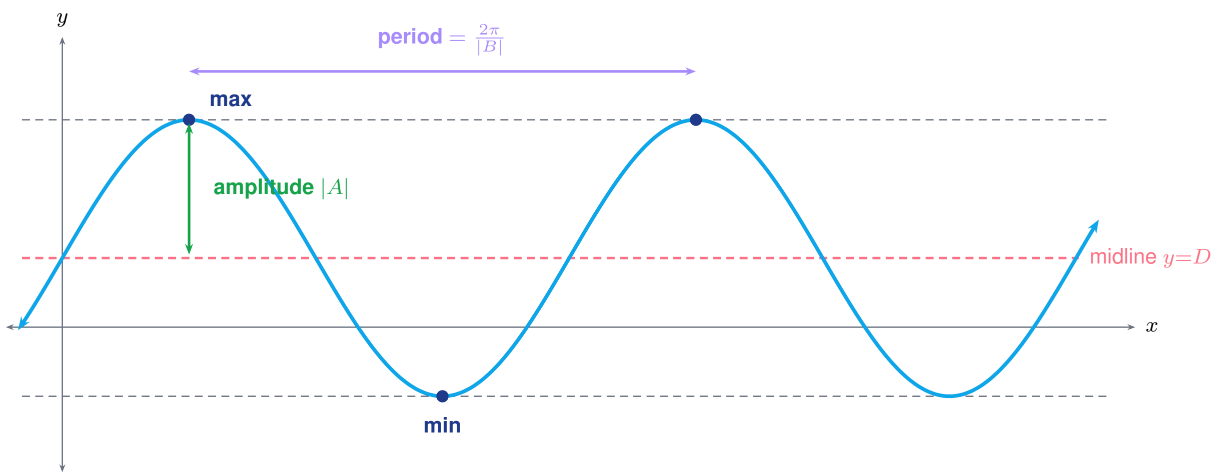
Choosing a trig tool

Use the unit circle for exact values and signs. Use the graph form for amplitude, period, shifts, and midline. Use the Law of Sines or Cosines when a triangle is not right.



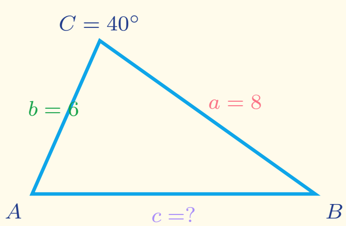
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
$c^2 = a^2 + b^2 - 2ab\cos C$

Anatomy of a sinusoid: $y = A\sin\big(B(x - C)\big) + D$ Amplitude $|A|$ is the height from the midline to a peak; the midline is $y = D$; the period $\frac{2\pi}{|B|}$ is one full cycle; C slides the wave sideways.



Solving a non-right triangle

Example



Two sides and the included angle are known, so start with the **Law of Cosines**:

$c^2 = a^2 + b^2 - 2ab\cos C = 8^2 + 6^2 - 2(8)(6)\cos 40°$

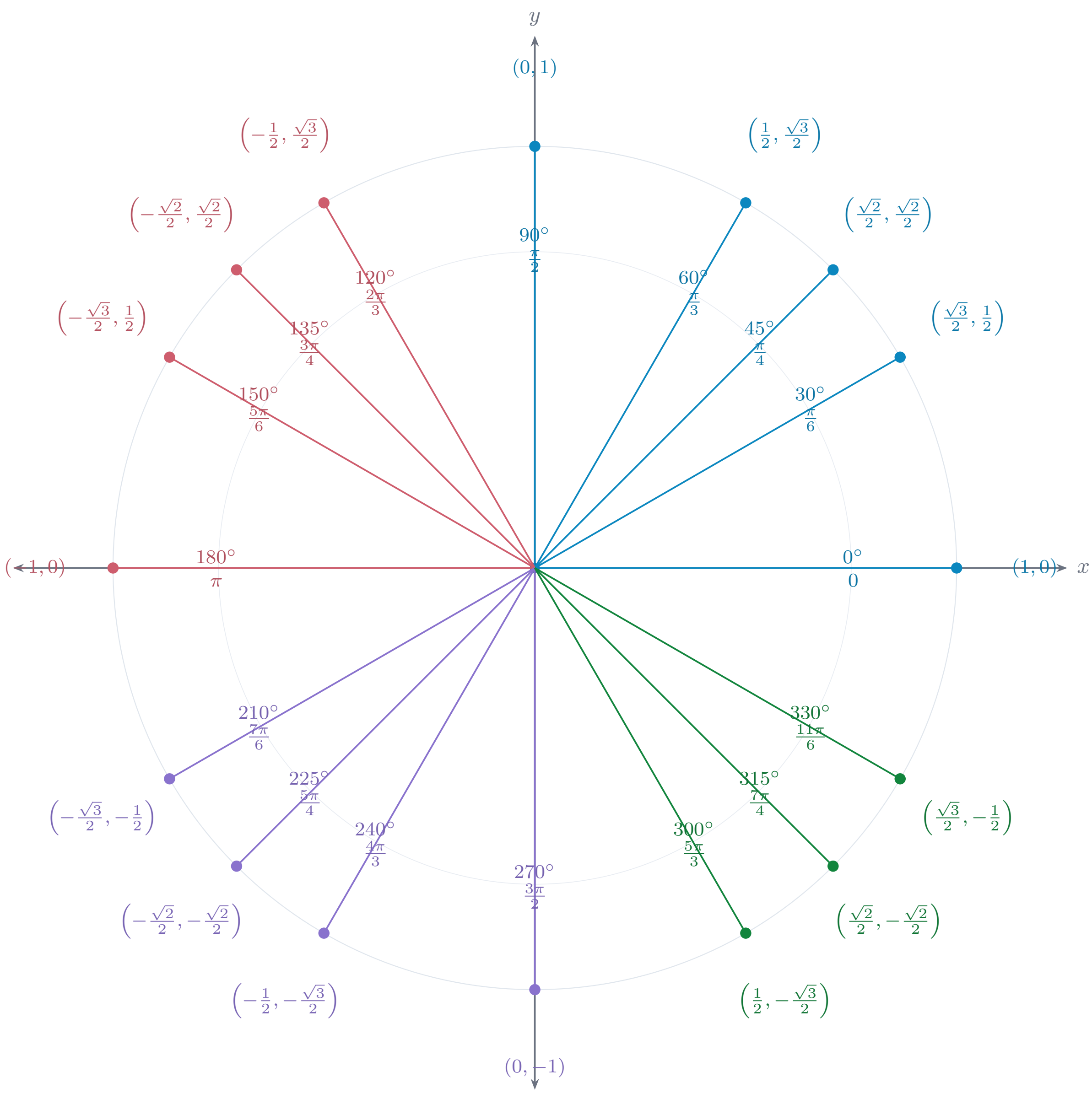
$c^2 = 100 - 96(0.766) \approx 26.5$, so $c \approx 5.15$.

Then the **Law of Sines** finds an angle: $\frac{\sin A}{8} = \frac{\sin 40°}{5.15} \Rightarrow A \approx 87°$.

12 Trigonometry Reference: Unit Circle & Identities

The Unit Circle

Every special angle in one place: each point is $(\cos\theta, \sin\theta)$. Read the angle in degrees and radians inside, and its coordinates outside.



QUADRANT SIGNS

In quadrant I all ratios are positive. After that, only one family stays positive: **S**ine in II, **T**angent in III, **C**osine in IV ("All Students Take Calculus").

Key trigonometric identities

Pythagorean identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Double-angle

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$
$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Sum & difference

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Half-angle

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$
$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

Even & odd

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

Cofunction & reciprocal

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \tan\left(\tfrac{\pi}{2} - \theta\right) = \cot\theta$$
$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$

Inverse trigonometric functions

$y = \sin^{-1} x$	Domain $-1 \le x \le 1$, Range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. $\sin(\sin^{-1} x) = x$.
$y = \cos^{-1} x$	Domain $-1 \le x \le 1$, Range $0 \le y \le \pi$. $\cos(\cos^{-1} x) = x$.
$y = \tan^{-1} x$	Domain all real x, Range $-\frac{\pi}{2} < y < \frac{\pi}{2}$. $\tan(\tan^{-1} x) = x$.
Watch out	$\sin^{-1} x$ is an *angle*, not $\frac{1}{\sin x}$. The reciprocal is $(\sin x)^{-1} = \csc x$.

Tutor's Note

You only need to memorize a few of these. The Pythagorean and reciprocal identities come straight from the unit circle, and the double-angle formulas are just the sum formulas with $A = B$. Derive the rest when you need them.

VERIFYING

To prove an identity, work on the more complicated side only and rewrite everything in terms of sin and cos. Do not move terms across the equals sign as if solving an equation.

13 Matrices

Matrix operations

Dimensions: An $m \times n$ matrix has m rows and n columns.

Addition/subtraction: Only matrices with the same dimensions can be added or subtracted.

Scalar multiplication: Multiply every entry by the scalar.

Matrix multiplication: AB is defined when columns of A equal rows of B. The result has rows of A and columns of B.

2×2 determinant: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$

2×2 inverse: $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if $ad - bc \neq 0$.

Matrix equation: $AX = B$ gives $X = A^{-1}B$ when A^{-1} exists.

Tutor's Note

Matrix multiplication is not entry-by-entry. Each entry in the product comes from a row times a column. Also, order matters: usually $AB \neq BA$.

Example

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2$, so the matrix has an inverse.

INVERSE

A matrix with determinant 0 is singular, so it has no inverse.

How matrix multiplication works

Each entry of AB is a *row of A* dotted with a *column of B*: multiply matching pieces, then add.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}_{B} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$(1)(5) + (2)(7) = 19$

Row 1 of A meets column 1 of B. Inner dimensions must match; the answer keeps the outer dimensions.