Z-Scores and the Normal Distribution

Name: ____________	Date: ____________	Score: ______ / 35

Quick Review

A **z-score** measures *how many standard deviations* a value is from the mean: $z = \frac{x-\mu}{\sigma}$. A z-score of $+1.5$ means "1.5 standard deviations above the mean"; a z-score of -2 means "2 standard deviations below."

Why standardize? Raw scores from different tests (SAT, ACT, a class quiz) live on different scales – comparing them directly is meaningless. Convert both to z-scores and you can compare apples to apples.

The standard normal distribution is $N(\mu = 0,\ \sigma = 1)$ – a bell curve centered at 0 with standard deviation 1. Any normal distribution can be *standardized* into $N(0, 1)$ by computing z-scores.

The empirical rule (68-95-99.7). For an approximately normal distribution: about 68% of values fall within $\pm 1\sigma$ of μ; about 95% within $\pm 2\sigma$; about 99.7% within $\pm 3\sigma$.

Reversing the formula. Given a z-score, recover the raw value with $x = \mu + z\sigma$. Quick check: $\mu = 500,\ \sigma = 100,\ z = 1.5 \Rightarrow x = 500 + 1.5(100) = 650$.

Common slips. Subtracting in the wrong order: it's $x-\mu$, not $\mu-x$. Forgetting to divide by σ (reporting a raw difference instead of a z-score). Confusing the mean (0) and standard deviation (1) of the standard normal. Using the empirical rule on a clearly non-normal distribution (it's only for bell-shaped ones).

PRACTICE

Convert between raw scores and z-scores; apply the empirical rule.

1. Formula for z-score ________

2. Use row 1 of the table to compute the z-score. ________

Row	μ	σ	x
1	70	8	86
2	500	100	400
3	1000	200	1400

3. Use row 2 of the table above to compute the z-score. ________

Row	μ	σ	x
1	70	8	86
2	500	100	400
3	1000	200	1400

4. Use row 3 of the table above to compute the z-score. ________

Row	μ	σ	x
1	70	8	86
2	500	100	400
3	1000	200	1400

5. Standard normal: mean and std dev ________

6. $z = -1.5$ means the value is ________

7. $\mu = 500, \sigma = 100, z = 1.5$. Find x. ________

8. $\mu = 70, \sigma = 10, z = -0.5$. Find x. ________

9. Empirical rule: % within $\pm 1\sigma$ ________

10. Empirical rule: % within $\pm 2\sigma$ ________

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11. Empirical rule: % within $\pm 3\sigma$ ________
12. Heights $N(70, 3)$. % between 64 and 76? ________
13. Student A: $z = 1$. Student B: $z = 0.5$. Who did better relative to their class? ________
14. Test $\mu = 80, \sigma = 5$. Score 90: $z = ?$ ________
15. Different test $\mu = 85, \sigma = 10$. Score 90: $z = ?$ ________
16. $z = 0$ means ________
17. Can a z-score be negative? ________
18. SAT $\mu = 1000, \sigma = 200$. Score 1500: $z = ?$ ________
19. IQ $N(100, 15)$. IQ 130 is how many SDs above mean? ________
20. If $z = 2$, what % of a normal distribution scores below? ________

Word Problems

21. On a standardized test with mean $\mu = 500$ and standard deviation $\sigma = 100$, a student scores 650. What is the student's z-score? What does that z-score say about the student's performance? ____
22. Heights of adult males are approximately normally distributed with mean 70 inches and standard deviation 3 inches. Approximately what percentage of adult males are between 64 and 76 inches tall? Use the empirical rule. ____
23. Student A scores 85 on a quiz with $\mu = 80$, $\sigma = 5$. Student B scores 90 on a different quiz with $\mu = 85$, $\sigma = 10$. Who did better relative to their class? ____
24. A factory produces light bulbs with a mean lifespan of $\mu = 1200$ hours and a standard deviation of $\sigma = 80$ hours. Lifespans are approximately normal. Approximately what percentage of bulbs last longer than 1360 hours? Use the empirical rule. ____

Additional Practice

25. Find the mean of $4, 6, 8, 10$. ________
26. Find the median of $3, 9, 4, 10, 7$. ________
27. Find the range of $12, 5, 9, 20$. ________
28. Find the mode of $2, 3, 3, 5, 8$. ________
29. Find z for $x = 72$, mean 60, standard deviation 6. ________
30. Interpret $z = -1.5$. ________
31. Predicted y for $\hat{y} = 2x + 5$ at $x = 6$. ________
32. Residual if actual $= 20$ and predicted $= 17$. ________
33. Positive association: slope sign? ________
34. Margin of error $= 3\%$ around 58%. ________
35. Sample or census: survey every student. ________

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Answer Keys

1. $z = \dfrac{x-\mu}{\sigma}$
2. 2
3. -1
4. 2
5. $\mu = 0, \sigma = 1$
6. 1.5σ below μ
7. 650
8. 65
9. 68%
10. 95%
11. 99.7%
12. 95%
13. Student A
14. 2
15. 0.5
16. $x = \mu$
17. Yes
18. 2.5
19. 2
20. $\approx 97.5\%$
21. $z = 1.5;$ about 93rd percentile
22. $\approx 95\%$
23. Student A ($z_A = 1 > z_B = 0.5$)
24. $\approx 2.5\%$

Additional Practice Answers

25. 7
26. 7
27. 15
28. 3
29. 2
30. 1.5 SD below mean
31. 17
32. 3
33. positive
34. 55% to 61%
35. census

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. A careful way to see it: Value minus mean, divided by standard deviation. Order matters: $x - \mu$, not $\mu - x$. That gives a quick check on the answer.

2. Keep the rule visible: Row 1: $z = \dfrac{86-70}{8} = \dfrac{16}{8} = 2$ – two standard deviations above the mean. That gives a quick check on the answer.

3. Row 2: $z = \dfrac{400-500}{100} = -1$ – one std dev below the mean. The negative sign matters.

4. Start with the key idea: Row 3: $z = \dfrac{1400-1000}{200} = \dfrac{400}{200} = 2$ – two std devs above. That gives a quick check on the answer.

5. A careful way to see it: The reference distribution: $N(0,1)$. All z-scores live on this scale. That gives a quick check on the answer.

6. Keep the rule visible: Negative $\Rightarrow$ below the mean. Magnitude $1.5 \Rightarrow 1.5$ standard deviations away. That gives a quick check on the answer.

7. Reverse the z-score formula to recover the raw value: $x = \mu + z\sigma$. Plug in $500 + 1.5(100) = 500 + 150 = 650$. A positive z lands you above the mean, which is why the answer is bigger than 500.

8. Use $x = \mu + z\sigma = 70 + (-0.5)(10) = 70 - 5 = 65$. Keep the negative sign on z – it's what pulls the value half a standard deviation below the mean instead of above it.

9. A careful way to see it: 68% of a normal distribution sits within one std dev of the mean. That gives a quick check on the answer.

10. Keep the rule visible: 95% of a normal distribution sits within two std devs. That gives a quick check on the answer.

11. One steady path is: 99.7% within three std devs – almost all of the distribution. That gives a quick check on the answer.

12. Start with the key idea: $64 = 70-2(3)$ and $76 = 70+2(3)$. Range is $\mu \pm 2\sigma$, so $\approx 95\%$ by the empirical rule. That gives a quick check on the answer.

13. Higher z-score $=$ better relative performance. Doesn't matter what the raw scores were – only the z-scores.

14. Apply $z = \dfrac{x-\mu}{\sigma} = \dfrac{90-80}{5} = \dfrac{10}{5} = 2$. Subtract the mean first ($x - \mu$, not $\mu - x$), then divide by σ. A score of 90 sits two standard deviations above the class mean.

15. One steady path is: $z = \dfrac{90-85}{10} = 0.5$. Same raw score (90), but on this test it's only half a std dev above the mean. That gives a quick check on the answer.

16. Start with the key idea: $z = 0$ tells you the value equals the mean exactly. This is the part to check before moving on, because it keeps the answer tied to the original question.

17. A careful way to see it: Negative z-scores indicate values *below* the mean. Standard, not an error. That gives a quick check on the answer.

18. Keep the rule visible: $z = \dfrac{1500-1000}{200} = \dfrac{500}{200} = 2.5$. Two and a half std devs above – impressive. That gives a quick check on the answer.

19. One steady path is: $z = \dfrac{130-100}{15} = \dfrac{30}{15} = 2$. Roughly the 97.5th percentile under a normal model. That gives a quick check on the answer.

20. By the empirical rule, 95% falls within $\pm 2\sigma$. By symmetry, 2.5% falls below -2σ, and 2.5% above $+2\sigma$. So 97.5% of values are below $z = 2$.

21. Apply the formula: $z = \dfrac{x-\mu}{\sigma} = \dfrac{650-500}{100} = \dfrac{150}{100} = 1.5$. The student is 1.5 standard deviations above the mean. Under a normal model, about 93% of test-takers score below this – a strong performance. (The empirical rule gives us 84% for $z = 1$ and 97.5% for $z = 2$; $z = 1.5$ lands between, around the 93rd percentile.)

22. Convert the endpoints to z-scores: $64 = 70 - 2(3) \Rightarrow z = -2$ and $76 = 70 + 2(3) \Rightarrow z = 2$. The interval is $\mu \pm 2\sigma$. By the empirical rule, $\approx 95\%$ of a normal distribution falls within two standard deviations of the mean. So about 95% of adult males are between 64 and 76 inches tall.

23. Convert each to a z-score so they live on a common scale. Student A: $z_A = \dfrac{85-80}{5} = 1$. Student B: $z_B = \dfrac{90-85}{10} = 0.5$. Even though Student B's raw score (90) is higher, Student A is further above their own class's mean. Relative to their respective classes, Student A did better. (Raw-score comparisons across different tests are misleading – standardize first.)

24. Start with the key idea: $1360 = 1200 + 2(80) \Rightarrow z = 2$. We want the percentage of bulbs with $z > 2$ (lifespan above 1360 hours). The empirical rule says 95% of values lie between $z = -2$ and $z = +2$, leaving 5% outside. By symmetry, half of that (2.5%) sits above $z = 2$. So about 2.5% of bulbs outlast 1360 hours – a small "elite tail" of unusually long-lasting bulbs. That gives a quick check on the answer.

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