Solving Trigonometric Equations

Name: ______________ Date: ______________ Score: ______ / 29

Quick Review

Solving a trig equation is like solving any algebraic equation, with one extra layer: trig functions are *periodic*, so a single equation usually has infinitely many solutions. The problem will tell you what interval to report.

Step 1: Isolate the trig function. Get to $\sin x = c$ (or cos, tan, etc.) before doing anything else. Move constants, divide, factor – normal algebra.

Step 2: Find the reference angle. Pretend the equation is positive; ask what acute angle gives that value. Memorized specials: $\sin x = \frac{1}{2} \Rightarrow$ ref $\frac{\pi}{6}$; $\sin x = \frac{\sqrt{2}}{2} \Rightarrow$ ref $\frac{\pi}{4}$; $\sin x = \frac{\sqrt{3}}{2} \Rightarrow$ ref $\frac{\pi}{3}$.

Step 3: Place the solutions in the right quadrants. Sine positive: Q1, Q2. Sine negative: Q3, Q4. Cosine positive: Q1, Q4. Cosine negative: Q2, Q3. Tangent positive: Q1, Q3. Tangent negative: Q2, Q4. (Memory aid: **ASTC** – All, Sine, Tangent, Cosine – starting in Q1 and going counterclockwise tells you which is positive in each quadrant.)

Step 4: Use the interval. If the problem asks for $[0, 2\pi)$, list every solution in that range. For a general solution, add $2\pi k$ (sine, cosine) or πk (tangent) for integer k.

Trig equations that look quadratic. If you see $2\sin^2 x - \sin x = 0$, factor: $\sin x(2\sin x - 1) = 0$. Then $\sin x = 0$ or $\sin x = \frac{1}{2}$.

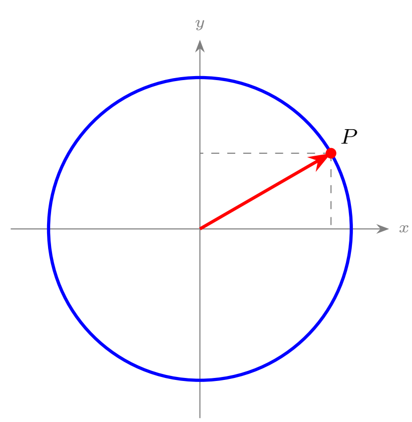
Scaled inputs. For $\sin(2x) = \frac{1}{2}$ on $[0, 2\pi)$, let $u = 2x$ so $u \in [0, 4\pi)$ – twice the usual range. Find all u first, then divide by 2 to get x. Don't lose half the solutions.

Watch out for. Dividing by a trig expression that could be zero – you'll lose solutions. Factor instead. After squaring, check for extraneous roots.

PRACTICE

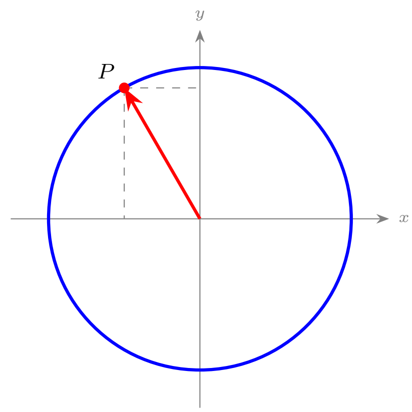
Find all solutions in $[0, 2\pi)$ unless told otherwise. Give exact answers using unit-circle angles.

1. Solve $\sin x = \frac{1}{2}$ on $[0, 2\pi)$. One solution is marked on the unit circle below; sine is positive, so find its partner in the other quadrant too. ________



2. Solve $\cos x = 0$ on $[0, 2\pi)$. ________
3. Solve $2\sin x - 1 = 0$ on $[0, 2\pi)$. ________

4. Solve $2\cos x + 1 = 0$ on $[0, 2\pi)$. First isolate the cosine; one of the two solution angles is shown on the circle below – find the second. ________



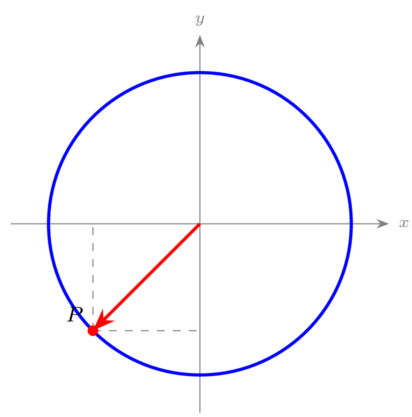
5. Solve $\tan x = 1$ on $[0, 2\pi)$. ________

6. Solve $\tan x = -1$ on $[0, 2\pi)$. ________

7. Solve $\sin x = 0$ on $[0, 2\pi)$. ________

8. Solve $\cos x = 1$ on $[0, 2\pi)$. ________

9. Solve $\sin x = -\dfrac{\sqrt{2}}{2}$ on $[0, 2\pi)$. The Q3 solution is marked below; sine is negative, so there is also a Q4 partner – find both. ________



10. Solve $2\sin^2 x - 1 = 0$ on $[0, 2\pi)$. ________

11. Solve $\cos^2 x = \sin^2 x$ on $[0, 2\pi)$. ________

12. Solve $\sin^2 x - \sin x = 0$ on $[0, 2\pi)$. ________

13. Solve $2\cos^2 x - 1 = 0$ on $[0, 2\pi)$. ________

14. Solve $\tan x = \sqrt{3}$ on $[0, 2\pi)$. ________

15. Solve $\sin x = 1$ on $[0, 2\pi)$. ________

16. Solve $\cos x = -1$ on $[0, 2\pi)$. ________

17. Solve $\sqrt{3}\tan x - 1 = 0$ on $[0, 2\pi)$. ________

18. Solve $\sin(2x) = \dfrac{1}{2}$ on $[0, 2\pi)$. ________

19. Solve $2\sin x \cos x = 0$ on $[0, 2\pi)$. ________

20. Solve $2\sin^2 x + \sin x - 1 = 0$ on $[0, 2\pi)$. ________

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Word Problems

21. A pendulum's angle from vertical is modeled by $\theta(t) = 0.5\sin(t)$ radians, with t in seconds. For what times in $[0, 2\pi)$ is the angle exactly 0.25 rad? ____

22. A daily-temperature model is $T(t) = 70 + 10\cos t$, where T is degrees Fahrenheit and t is in radians ($[0, 2\pi)$ represents one day). For what times in $[0, 2\pi)$ does $T(t) = 65°$? ____

23. A buoy's vertical position is $y(t) = 2\sin t$ meters. How many times does the buoy reach its maximum height of 2 meters during the interval $[0, 4\pi)$? ____

24. An AC voltage signal is $V(t) = 120\sin(t)$ volts. Find all times in $[0, 2\pi)$ when $V(t) = -60\sqrt{3}$ volts. ____

Additional Practice

25. Find $\sin\theta$ if opposite $= 5$, hypotenuse $= 13$. ________

26. Find $\cos\theta$ if adjacent $= 12$, hypotenuse $= 13$. ________

27. Find $\tan\theta$ if opposite $= 7$, adjacent $= 4$. ________

28. Find $\sin 30°$. ________

29. Find $\cos 60°$. ________

Answer Keys

1. $x = \frac{\pi}{6}, \frac{5\pi}{6}$
2. $x = \frac{\pi}{2}, \frac{3\pi}{2}$
3. $x = \frac{\pi}{6}, \frac{5\pi}{6}$
4. $x = \frac{2\pi}{3}, \frac{4\pi}{3}$
5. $x = \frac{\pi}{4}, \frac{5\pi}{4}$
6. $x = \frac{3\pi}{4}, \frac{7\pi}{4}$
7. $x = 0, \pi$
8. $x = 0$
9. $x = \frac{5\pi}{4}, \frac{7\pi}{4}$
10. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
11. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
12. $x = 0, \frac{\pi}{2}, \pi$
13. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
14. $x = \frac{\pi}{3}, \frac{4\pi}{3}$
15. $x = \frac{\pi}{2}$
16. $x = \pi$
17. $x = \frac{\pi}{6}, \frac{7\pi}{6}$
18. $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
19. $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$
20. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$
21. $t = \frac{\pi}{6}, \frac{5\pi}{6}$
22. $t = \frac{2\pi}{3}, \frac{4\pi}{3}$
23. 2 times
24. $t = \frac{4\pi}{3}, \frac{5\pi}{3}$

Additional Practice Answers

25. $\frac{5}{13}$
26. $\frac{12}{13}$
27. $\frac{7}{4}$
28. $\frac{1}{2}$
29. $\frac{1}{2}$

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. The reference angle whose sine is $\frac{1}{2}$ is $\frac{\pi}{6}$. Since $\sin x$ is positive, x lives where the y-coordinate is positive – Q1 and Q2 (the marked point and its mirror across the y-axis). That gives $x = \frac{\pi}{6}$ in Q1 and $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$ in Q2. Don't stop at the first solution – the interval $[0, 2\pi)$ holds both.

2. Cosine is the x-coordinate on the unit circle; it equals zero at the top and bottom: $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

3. One steady path is: Isolate first: $\sin x = \frac{1}{2}$. Same as the first problem – $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. That gives a quick check on the answer.

4. Isolate first: $2\cos x + 1 = 0$ gives $\cos x = -\frac{1}{2}$. The reference angle is $\frac{\pi}{3}$. Cosine (the x-coordinate) is negative on the left side of the circle – Q2 and Q3 – so $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (the marked Q2 point) and $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ in Q3.

5. A careful way to see it: Tangent is positive in Q1 and Q3; reference $\frac{\pi}{4}$. So $x = \frac{\pi}{4}$ and $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$. That gives a quick check on the answer.

6. Tangent is negative in Q2 and Q4; reference $\frac{\pi}{4}$. $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ and $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

7. Sine is zero on the x-axis: at $x = 0$ (start) and $x = \pi$ (halfway). Don't include 2π – the interval is half-open.

8. Cosine maxes out at 1 only at $x = 0$ (in the interval $[0, 2\pi)$). 2π would also work, but it's excluded.

9. The reference angle for $\sin = \frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$. Sine is negative where the y-coordinate dips below the axis – the bottom half, Q3 and Q4. So $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ (the marked Q3 point) and $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ in Q4. A common slip is reporting the Q3 angle alone and forgetting its Q4 partner.

10. Keep the rule visible: $\sin^2 x = \frac{1}{2} \Rightarrow \sin x = \pm\frac{\sqrt{2}}{2}$. All four reference-$\frac{\pi}{4}$ angles work because the sign on both branches is covered. That gives a quick check on the answer.

11. One steady path is: $|\cos x| = |\sin x|$, so the reference angle is $\frac{\pi}{4}$ and every quadrant supplies a solution. (Or: divide both sides by $\cos^2 x$ to get $\tan^2 x = 1$.) That gives a quick check on the answer.

12. Factor: $\sin x(\sin x - 1) = 0$. So $\sin x = 0 \Rightarrow x = 0, \pi$ or $\sin x = 1 \Rightarrow x = \frac{\pi}{2}$. Three solutions. (Don't divide both sides by $\sin x$ – you'd lose $x = 0, \pi$.)

13. A careful way to see it: $\cos^2 x = \frac{1}{2} \Rightarrow \cos x = \pm\frac{\sqrt{2}}{2}$. All four $\frac{\pi}{4}$-reference angles. This is the part to check before moving on, because it keeps the answer tied to the original question.

14. Keep the rule visible: Tangent positive: Q1 and Q3. Reference $\frac{\pi}{3}$. So $x = \frac{\pi}{3}$ and $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$. That gives a quick check on the answer.

15. One steady path is: Sine reaches its max of 1 only at $x = \frac{\pi}{2}$. This is the part to check before moving on, because it keeps the answer tied to the original question.

16. Start with the key idea: Cosine bottoms out at -1 only at $x = \pi$. This is the part to check before moving on, because it keeps the answer tied to the original question.

17. Isolate: $\tan x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. Reference $\frac{\pi}{6}$; tangent positive in Q1, Q3. So $x = \frac{\pi}{6}$ and $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

18. Let $u = 2x$, so $u \in [0, 4\pi)$ – two full periods. $\sin u = \frac{1}{2}$ at $u =$

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$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$. Divide each by 2 to get x.

19. Zero product: $\sin x = 0$ gives $x = 0, \pi$; $\cos x = 0$ gives $x = \frac{\pi}{2}, \frac{3\pi}{2}$. Four solutions total. (Bonus: $2\sin x\cos x = \sin 2x$, so this is just $\sin 2x = 0$.)

20. Factor like a quadratic in $\sin x$: $(2\sin x - 1)(\sin x + 1) = 0$. Then $\sin x = \frac{1}{2}$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$, and $\sin x = -1$ gives $x = \frac{3\pi}{2}$.

21. Set $0.5\sin t = 0.25$, so $\sin t = \frac{1}{2}$. Solutions in $[0, 2\pi)$: $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. (The pendulum passes through 0.25 rad twice in one full cycle – once going each way.)

22. Keep the rule visible: $70 + 10\cos t = 65 \Rightarrow \cos t = -\frac{1}{2}$. Cosine is $-\frac{1}{2}$ in Q2 and Q3, with reference $\frac{\pi}{3}$: $t = \frac{2\pi}{3}$ and $t = \frac{4\pi}{3}$. That gives a quick check on the answer.

23. One steady path is: $2\sin t = 2 \Rightarrow \sin t = 1$, which happens only at $t = \frac{\pi}{2} + 2\pi k$. In $[0, 4\pi)$ that's $t = \frac{\pi}{2}$ and $t = \frac{5\pi}{2}$ – two times. (One per full period of length 2π, and the interval is two periods long.) That gives a quick check on the answer.

24. Start with the key idea: $120\sin t = -60\sqrt{3} \Rightarrow \sin t = -\frac{\sqrt{3}}{2}$. Sine is negative in Q3 and Q4; reference angle $\frac{\pi}{3}$. So $t = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$ (Q3) and $t = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ (Q4). That gives a quick check on the answer.