Inverse Trigonometric Functions

Name: ______________________	Date: ______________	Score: _____ / 29

Quick Review

Inverse trig functions answer the question "which angle has this trig value?" But sine, cosine, and tangent each hit the same output infinitely often, so we have to restrict their inputs to get a clean one-to-one piece – the **principal range**.

The three principal ranges (memorize these).

$\sin^{-1} x$ outputs angles in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$\cos^{-1} x$ outputs angles in $[0, \pi]$.

$\tan^{-1} x$ outputs angles in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Notation. $\sin^{-1} x$ and $\arcsin x$ are the same thing – the inverse function, *not* the reciprocal. ($\csc x$ is the reciprocal of $\sin x$; never use $\sin^{-1} x$ for that.)

Input domains. Since the original sine/cosine produce values in $[-1, 1]$, the inputs to $\sin^{-1}$ and $\cos^{-1}$ must stay inside $[-1, 1]$. Asking for $\arcsin(1.2)$ is asking for an angle whose sine is 1.2 – there isn't one. Arctangent is friendlier: every real number is a valid tangent value, so $\tan^{-1} x$ accepts all of $\mathbb{R}$.

Composition shortcuts. For x in the input domain of the inverse, $\sin(\sin^{-1} x) = x$, $\cos(\cos^{-1} x) = x$, $\tan(\tan^{-1} x) = x$. The other direction (inverse-of-direct) only returns x when x is already in the principal range – otherwise you get a reduced equivalent.

Right-triangle trick for compositions. For something like $\sin(\cos^{-1}(3/5))$, draw a right triangle with the cosine ratio $3/5$ (adjacent 3, hypotenuse 5). Pythagoras gives the third side as 4, so $\sin = 4/5$. Check the quadrant from the principal range: $\cos^{-1}$ outputs $[0, \pi]$, where sine is non-negative, so the answer is positive $4/5$.

Common slips. Confusing $\sin^{-1} x$ with $\frac{1}{\sin x}$. Forgetting that $\arccos$ outputs $[0, \pi]$ (not the same range as $\arcsin$). Plugging $\arcsin(1.5)$ into a calculator and reporting the error code – the right answer is "undefined."

PRACTICE

Give exact values where possible; otherwise round to two decimals. Inputs outside the domain should be marked undefined.

1. Evaluate $\arcsin(0)$. ________
2. Evaluate $\arcsin\left(\frac{1}{2}\right)$. ________
3. Evaluate $\arccos(0)$. ________
4. Evaluate $\arccos(1)$. ________
5. Evaluate $\arccos(-1)$. ________
6. Evaluate $\arctan(1)$. ________
7. Evaluate $\arctan(-1)$. ________
8. Evaluate $\arcsin\left(\frac{\sqrt{3}}{2}\right)$. ________
9. Evaluate $\arccos\left(-\frac{1}{2}\right)$. ________
10. Evaluate $\arctan(0)$. ________

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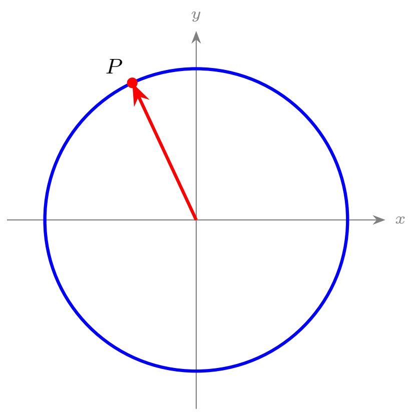
11. State the principal range of $\arccos$. The output angle of $\arccos$ always lands somewhere along the upper arc shown (here a sample terminal point P); give that range in interval form.

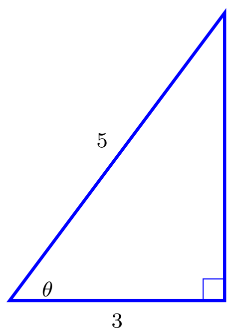


12. State the principal range of $\arctan$.

13. Is $\arcsin\left(\frac{3}{2}\right)$ defined? If so, find it.

14. Evaluate $\sin(\arcsin(0.4))$.

15. Evaluate $\sin\left(\arccos\left(\frac{3}{5}\right)\right)$. Let $\theta = \arccos\frac{3}{5}$ be the angle in the right triangle below, then read off its sine.

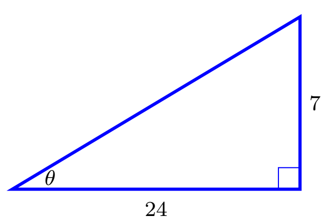


16. Evaluate $\arcsin(-1)$.

17. Evaluate $\arctan(\sqrt{3})$.

18. Evaluate $\cos(\arcsin(0))$.

19. Let $\theta = \arctan\left(\frac{7}{24}\right)$. Find $\sin\theta$. The triangle below has $\tan\theta = \frac{7}{24}$ (opposite over adjacent).



20. Evaluate $\arcsin\left(\sin\frac{3\pi}{4}\right)$.

Word Problems

21. A wheelchair ramp rises 1 foot over a horizontal run of 12 feet. What angle does the ramp make with the ground? Round to the nearest tenth of a degree. ____

22. A boat travels 40 km east and then 30 km north. From the starting point, what is the bearing of the boat's final position measured east of north? Round to the nearest tenth of a degree. ____

23. A pendulum swings so that the sine of the angle from vertical equals 0.25 at its maximum displacement. Find the angle of maximum displacement, in radians, rounded to three decimal places. ____

24. Astronomers observe a star whose elevation above the horizon has cosine equal to $\frac{4}{5}$. What is the sine of the elevation angle, assuming the angle is between 0 and $\frac{\pi}{2}$? ____

Additional Practice

25. Find $\sin\theta$ if opposite $= 5$, hypotenuse $= 13$. ________

26. Find $\cos\theta$ if adjacent $= 12$, hypotenuse $= 13$. ________

27. Find $\tan\theta$ if opposite $= 7$, adjacent $= 4$. ________

28. Find $\sin 30°$. ________

29. Find $\cos 60°$. ________

Answer Keys

1. $\boxed{0}$
2. $\boxed{\frac{\pi}{6}}$
3. $\boxed{\frac{\pi}{2}}$
4. $\boxed{0}$
5. $\boxed{\pi}$
6. $\boxed{\frac{\pi}{4}}$
7. $\boxed{-\frac{\pi}{4}}$
8. $\boxed{\frac{\pi}{3}}$
9. $\boxed{\frac{2\pi}{3}}$
10. $\boxed{0}$
11. $\boxed{[0, \pi]}$
12. $\boxed{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$
13. Undefined
14. $\boxed{0.4}$
15. $\boxed{\frac{4}{5}}$
16. $\boxed{-\frac{\pi}{2}}$
17. $\boxed{\frac{\pi}{3}}$
18. $\boxed{1}$
19. $\boxed{\frac{7}{25}}$
20. $\boxed{\frac{\pi}{4}}$
21. $\boxed{\approx 4.8^\circ}$
22. $\boxed{\approx 53.1^\circ}$
23. $\boxed{\approx 0.253 \text{ rad}}$
24. $\boxed{\frac{3}{5}}$

Additional Practice Answers

25. $\boxed{\frac{5}{13}}$
26. $\boxed{\frac{12}{13}}$
27. $\boxed{\frac{7}{4}}$
28. $\boxed{\frac{1}{2}}$
29. $\boxed{\frac{1}{2}}$

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. A careful way to see it: $\sin 0 = 0$, and 0 is inside the principal range $[-\pi/2, \pi/2]$. So $\arcsin(0) = 0$. That gives a quick check on the answer.

2. Keep the rule visible: $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6}$ is in the principal range, so the answer is $\frac{\pi}{6}$. (The other angle with sine $\frac{1}{2}$ is $\frac{5\pi}{6}$ – valid for sine, but outside the arcsin principal range.) That gives a quick check on the answer.

3. One steady path is: $\cos\frac{\pi}{2} = 0$ and $\frac{\pi}{2} \in [0, \pi]$, so $\arccos(0) = \frac{\pi}{2}$. This is the part to check before moving on, because it keeps the answer tied to the original question.

4. Start with the key idea: $\cos 0 = 1$, and 0 is in the principal range $[0, \pi]$. Done. This is the part to check before moving on, because it keeps the answer tied to the original question.

5. A careful way to see it: $\cos\pi = -1$, and π is the right endpoint of $[0, \pi]$ – so it's in the principal range. Answer is π. That gives a quick check on the answer.

6. Keep the rule visible: $\tan\frac{\pi}{4} = 1$ and $\frac{\pi}{4}$ is in $(-\pi/2, \pi/2)$. This is the part to check before moving on, because it keeps the answer tied to the original question.

7. Tangent is odd, so the negative input gives a negative angle. $\tan(-\pi/4) = -1$ and $-\pi/4$ sits in the principal range.

8. Start with the key idea: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $[-\pi/2, \pi/2]$. This is the part to check before moving on, because it keeps the answer tied to the original question.

9. We need an angle in $[0, \pi]$ with cosine $-\frac{1}{2}$. That's $\frac{2\pi}{3}$ (Q2, where cosine is negative).

10. Keep the rule visible: $\tan 0 = 0$ and 0 sits at the center of the principal range. This is the part to check before moving on, because it keeps the answer tied to the original question.

11. Cosine repeats, so to make it invertible we keep only the top half of the circle, where θ runs from 0 to π – that is the slice P rides along. On that arc each cosine value appears exactly once, so $\arccos$ outputs angles in $[0, \pi]$. (Don't confuse this with arcsin's range $[-\pi/2, \pi/2]$.)

12. Open interval – tangent never actually reaches $\pm\pi/2$ because it has asymptotes there.

13. A careful way to see it: The domain of $\arcsin$ is $[-1, 1]$. Since $\frac{3}{2} > 1$, no real angle has sine equal to $\frac{3}{2}$. That gives a quick check on the answer.

14. Keep the rule visible: Inside the domain, $\sin$ undoes $\arcsin$ directly: $\sin(\arcsin x) = x$ for $x \in [-1, 1]$. That gives a quick check on the answer.

15. Let $\theta = \arccos\frac{3}{5}$, so $\cos\theta = \frac{3}{5}$: adjacent 3, hypotenuse 5, as labeled. Pythagoras gives the opposite side $\sqrt{5^2 - 3^2} = \sqrt{16} = 4$, so the raw ratio for sine is $\frac{4}{5}$. Now the sign: $\arccos$ outputs angles in $[0, \pi]$ where sine is never negative, so $\sin\theta = +\frac{4}{5}$.

16. Start with the key idea: $\sin(-\pi/2) = -1$ and $-\pi/2$ is the lower endpoint of the principal range. That gives a quick check on the answer.

17. A careful way to see it: $\tan\frac{\pi}{3} = \sqrt{3}$, and $\frac{\pi}{3}$ sits in $(-\pi/2, \pi/2)$. This is the part to check before moving on, because it keeps the answer tied to the original question.

18. Keep the rule visible: $\arcsin(0) = 0$ and $\cos 0 = 1$. (Or: the right-triangle version – if $\sin\theta = 0$, then $\cos\theta = \pm 1$, and the arcsin principal range $[-\pi/2, \pi/2]$ keeps cosine non-negative.) That gives a quick check on the answer.

19. Since $\tan\theta = \frac{7}{24}$, label the triangle with opposite 7 and adjacent 24. The hypotenuse comes from Pythagoras: $\sqrt{7^2 + 24^2} = \sqrt{625} = 25$, so $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{7}{25}$. The sign is positive because $\arctan$ of a positive number lands in Q1 (its range is $(-\pi/2, \pi/2)$, and a positive tangent puts θ above the axis).

20. Start with the key idea: $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$. Then $\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ because $\frac{\pi}{4}$ is in $[-\pi/2, \pi/2]$. (Watch out: the answer is NOT $\frac{3\pi}{4}$ – that's outside the principal range.) That gives a quick check on the answer.

21. The tangent of the ramp angle equals $\frac{\text{rise}}{\text{run}} = \frac{1}{12}$. So $\theta = \arctan(1/12) \approx 4.76^\circ$, which rounds to 4.8°. (ADA guidance is about 4.76° max, so this ramp is just at the limit.)

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22. Picture the right triangle with legs 40 (east) and 30 (north). The bearing east of north uses the angle between north and the resultant: $\tan\theta = \frac{\text{east}}{\text{north}} = \frac{40}{30}$. So $\theta = \arctan(4/3) \approx 53.13°$, rounded to $53.1°$.

23. The angle is $\arcsin(0.25) \approx 0.2527$ rad. Rounded to three decimals: 0.253 rad. (A small angle, about $14.5°$ – a gentle swing.)

24. Set $\theta = \arccos(4/5)$, so $\theta \in [0, \pi/2]$ (it's positive, and arccos of a positive value stays in $[0, \pi/2]$). In a right triangle with adjacent 4 and hypotenuse 5, the opposite side is $\sqrt{25-16} = 3$. So $\sin\theta = \frac{3}{5}$. (Classic 3-4-5 triangle.)