Infinite Geometric Series

Name: ____________	Date: ____________	Score: ______ / 30

Quick Review

What happens to $a_1 + a_1 r + a_1 r^2 + \cdots$ when we never stop adding? Two possibilities, and the answer hinges on r.

Convergence condition. For a nonzero a_1, the infinite geometric series converges *if and only if* $|r| < 1$. The intuition: $r^n \to 0$ when $|r| < 1$, so the finite formula $\frac{a_1(1-r^n)}{1-r}$ approaches $\frac{a_1}{1-r}$ as $n \to \infty$. If $|r| \geq 1$, the terms don't shrink to zero – they stay big (or grow) and the sum runs off to infinity (or oscillates).

Sum formula (only when it converges). $S = \frac{a_1}{1-r}$.

The $|r| = 1$ edge. $r = 1$ adds a_1 forever (diverges to $\pm\infty$). $r = -1$ flips back and forth (no settled value). Either way, no convergence.

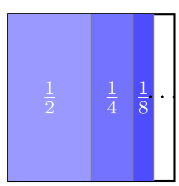
Repeating decimals. Any repeating decimal is an infinite geometric series in disguise. $0.\overline{3} = 0.3 + 0.03 + 0.003 + \cdots$ with $a_1 = 0.3$, $r = 0.1$. So $S = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{1}{3}$. $0.\overline{27}$ has $a_1 = 0.27$, $r = 0.01$, giving $\frac{0.27}{0.99} = \frac{27}{99} = \frac{3}{11}$.

Common slips. Using the formula when $|r| \geq 1$ – the answer is meaningless (the series diverges). Forgetting the absolute value: $r = -0.5$ converges (since $|-0.5| = 0.5 < 1$) but $r = -1.2$ diverges (since $|-1.2| = 1.2 > 1$). Mistaking a_1 for the first power of r – a_1 is the actual first term you see, including any leading constant.

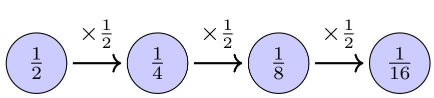
PRACTICE

Decide whether each infinite geometric series converges. If it does, find the sum using $S = a_1/(1-r)$.

1. Does $\sum_{k=0}^{\infty} 6 \cdot (\frac{1}{2})^k$ converge? If yes, find the sum. ________
2. Does $\sum_{k=0}^{\infty} 4 \cdot 2^k$ converge? ________
3. Write $0.\overline{3}$ as a fraction. ________
4. An infinite geometric series has $r = \frac{1}{4}$ and $S = 20$. Find a_1. ________
5. Find the sum of the shaded area shown. ________



6. Find $\sum_{k=0}^{\infty} 3(0.4)^k$. ________
7. Does $5 + 5(-1.2) + 5(-1.2)^2 + \cdots$ converge? ________
8. Find the sum: $12 - 6 + 3 - \frac{3}{2} + \cdots$ ________
9. Write $0.\overline{27}$ as a fraction in lowest terms. ________
10. Compute $\sum_{k=1}^{\infty} \frac{1}{2^k}$. ________



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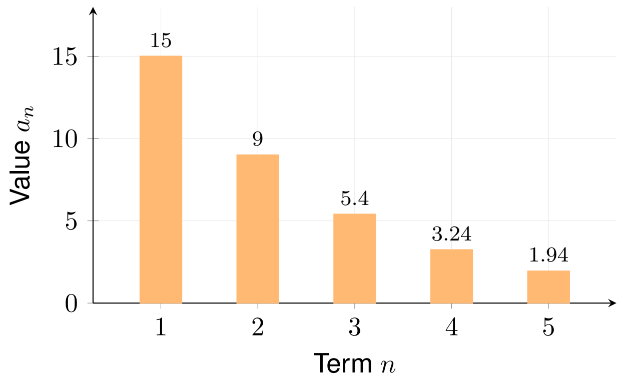
11. A ball rebounds to $\frac{3}{5}$ of its previous height. If the first rebound is 15 feet, find the total of all rebound heights. ________



12. True or False: $r = -1$ makes an infinite geometric series converge. ________

13. Find $\sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k$. ________

14. Write $0.\overline{6}$ as a fraction. ________

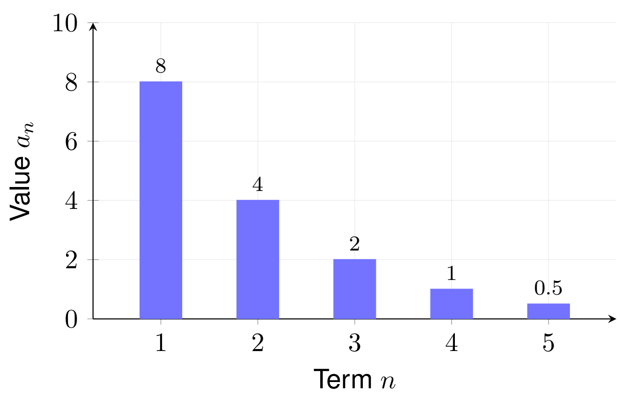
15. An infinite geometric series converges to 24 with $a_1 = 18$. Find r. ________

16. Find the sum: $9 + 3 + 1 + \frac{1}{3} + \cdots$ ________

17. Compute $\sum_{k=0}^{\infty} 5\left(-\frac{1}{2}\right)^k$. ________

18. True or False: every infinite geometric series with $a_1 > 0$ and $r > 0$ converges. ________

19. Find the bar-chart series sum: $a_1 = 8$, $r = \frac{1}{2}$. ________



20. A pendulum's arc length is 20 cm on the first swing, and each later swing is 80% of the previous arc. Find the total arc length over all swings (assuming it never stops). ________

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Word Problems

21. A super ball is dropped from a height of 9 feet. Each bounce reaches $\frac{2}{3}$ of the previous bounce's height. What is the total distance (up and down) the ball travels before coming to rest? ____

22. A factory's output drops by 10% each year due to aging equipment. If it produced $5{,}000$ widgets in year 1, find the total number of widgets produced over all future years. ____

23. Express $0.\overline{45}$ as a fraction in lowest terms using an infinite geometric series. ____

24. A drug-dosing model: a patient takes 80 mg daily, and the body retains 50% of the previous day's amount each day. The long-run level (after many days) is the sum of the residuals from all prior doses, plus today's new dose. Compute this long-run equilibrium dose. ____

Additional Practice

25. Find the next term: $4, 9, 14, 19, \ldots$. ____

26. Find a_{10} if $a_1 = 3$ and $d = 5$. ____

27. Find the next term: $2, 6, 18, 54, \ldots$. ____

28. Find a_6 if $a_1 = 5$ and $r = 2$. ____

29. Sum $1 + 2 + 3 + \cdots + 20$. ____

30. Find S_5 for $3, 6, 12, 24, 48$. ____

Answer Keys

1. converges to 12
2. No (diverges)
3. $\frac{1}{3}$
4. $a_1 = 15$
5. 1
6. 5
7. No (diverges)
8. 8
9. $\frac{3}{11}$
10. 1
11. 37.5 ft
12. False
13. $\frac{3}{4}$
14. $\frac{2}{3}$
15. $r = \frac{1}{4}$
16. $\frac{27}{2}$
17. $\frac{10}{3}$
18. False
19. 16
20. 100 cm
21. 45 ft
22. 50,000 widgets
23. $\frac{5}{11}$
24. 160 mg

Additional Practice Answers

25. 24
26. 48
27. 162
28. 160
29. 210
30. 93

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. A careful way to see it: $|r| = \frac{1}{2} < 1$, so it converges. $a_1 = 6$, so $S = \frac{6}{1-1/2} = \frac{6}{1/2} = 12$. That gives a quick check on the answer.

2. Keep the rule visible: $|r| = 2 \geq 1$. The terms $4, 8, 16, 32, \ldots$ blow up, so the sum runs to infinity. That gives a quick check on the answer.

3. One steady path is: $0.\overline{3} = 0.3+0.03+0.003+\cdots$. Geometric with $a_1 = 0.3$, $r = 0.1$. $S = \frac{0.3}{0.9} = \frac{1}{3}$. That gives a quick check on the answer.

4. Start with the key idea: $S = \frac{a_1}{1-r} = \frac{a_1}{3/4} = \frac{4a_1}{3} = 20$. So $a_1 = 20 \cdot \frac{3}{4} = 15$. That gives a quick check on the answer.

5. A careful way to see it: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ is geometric with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. $S = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$. (The picture makes it obvious – the shaded region fills the whole square.) That gives a quick check on the answer.

6. Keep the rule visible: $|0.4| < 1$, so converges. $S = \frac{3}{1-0.4} = \frac{3}{0.6} = 5$. This is the part to check before moving on, because it keeps the answer tied to the original question.

7. One steady path is: $|r| = 1.2 \geq 1$, so the series diverges. (The terms grow in magnitude while alternating sign – no settled value.) That gives a quick check on the answer.

8. Ratio: $-6/12 = -\frac{1}{2}$. $|r| = \frac{1}{2} < 1$, so it converges. $S = \frac{12}{1-(-1/2)} = \frac{12}{3/2} = 8$. (Negative r is fine – only $|r| < 1$ matters for convergence.)

9. A careful way to see it: $0.\overline{27} = 0.27+0.0027+\cdots$. Geometric with $a_1 = 0.27$, $r = 0.01$. $S = \frac{0.27}{0.99} = \frac{27}{99} = \frac{3}{11}$ (divide by 9). That gives a quick check on the answer.

10. Pull out $\frac{1}{2}$: $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}+\frac{1}{4}+\cdots$ with $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. $S = \frac{1/2}{1/2} = 1$. (Same as the shaded-square sum.)

11. One steady path is: $a_1 = 15$, $r = \frac{3}{5}$. $S = \frac{15}{1-3/5} = \frac{15}{2/5} = 15 \cdot \frac{5}{2} = \frac{75}{2} = 37.5$ feet. (The bars show how quickly the rebound heights shrink.) That gives a quick check on the answer.

12. Start with the key idea: $|-1| = 1$, which fails the strict condition $|r| < 1$. The partial sums oscillate between two values (e.g. $a_1, 0, a_1, 0, \ldots$ for $a_1 + (-a_1) + a_1 + \cdots$), never settling. That gives a quick check on the answer.

13. A careful way to see it: $a_1 = 1$ (since $(-\frac{1}{3})^0 = 1$), $r = -\frac{1}{3}$. $|r| = \frac{1}{3} < 1$, so converges. $S = \frac{1}{1-(-1/3)} = \frac{1}{4/3} = \frac{3}{4}$. That gives a quick check on the answer.

14. Keep the rule visible: $a_1 = 0.6$, $r = 0.1$. $S = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$. This is the part to check before moving on, because it keeps the answer tied to the original question.

15. One steady path is: $24 = \frac{18}{1-r}$, so $24(1-r) = 18 \Rightarrow 1-r = \frac{18}{24} = \frac{3}{4} \Rightarrow r = \frac{1}{4}$. (Quick check: $|r| = \frac{1}{4} < 1$ ✓.) That gives a quick check on the answer.

16. Start with the key idea: $r = 3/9 = \frac{1}{3}$. $S = \frac{9}{1-1/3} = \frac{9}{2/3} = \frac{27}{2}$. This is the part to check before moving on, because it keeps the answer tied to the original question.

17. A careful way to see it: $a_1 = 5$, $r = -\frac{1}{2}$. $|r| = \frac{1}{2} < 1$ ✓. $S = \frac{5}{1-(-1/2)} = \frac{5}{3/2} = \frac{10}{3}$. That gives a quick check on the answer.

18. Convergence depends on $|r| < 1$, not on the signs of a_1 or r. $a_1 = 2, r = 3$ gives $2+6+18+\cdots$ – positive and increasing, but it diverges.

19. One steady path is: $S = \frac{8}{1-1/2} = \frac{8}{1/2} = 16$. (The bar chart makes the geometric shrinkage visible – after term 5, there's almost nothing left to add.) That gives a quick check on the answer.

20. Start with the key idea: $a_1 = 20$, $r = 0.8$. $|r| < 1$ ✓. $S = \frac{20}{1-0.8} = \frac{20}{0.2} = 100$ cm. (Reality check: positive total, larger than the first swing – both expected.) That gives a quick check on the answer.

21. The first drop is 9 feet (down only). After that, each bounce goes up and then back down by the same height, so the total up-and-down contribution from the bounces is $2(9 \cdot \frac{2}{3} + 9 \cdot (\frac{2}{3})^2 + \cdots)$. The geometric sum inside is

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$\dfrac{9 \cdot 2/3}{1 - 2/3} = \dfrac{6}{1/3} = 18$. Doubling: $2 \cdot 18 = 36$. Total: $9 + 36 = 45$ feet. (Reality check: the ball falls 9 ft, then bounces small amounts forever – the total is finite because the bounces shrink geometrically.)

22. Keep the rule visible: $a_1 = 5{,}000$, $r = 0.9$ (each year is 90% of the previous). $|r| < 1$ ✓. $S = \dfrac{5{,}000}{1 - 0.9} = \dfrac{5{,}000}{0.1} = 50{,}000$ widgets. (Reality check: a 10% drop is slow decay, so the total is 10 times the first year – this matches the rule $S = a_1/(1 - r)$ when r is close to 1.) That gives a quick check on the answer.

23. Write $0.\overline{45} = 0.45 + 0.0045 + 0.000045 + \cdots$. Geometric with $a_1 = 0.45 = \frac{45}{100}$ and $r = 0.01 = \frac{1}{100}$. Sum: $S = \dfrac{45/100}{1 - 1/100} = \dfrac{45/100}{99/100} = \dfrac{45}{99} = \dfrac{5}{11}$ (divide top and bottom by 9).

24. At equilibrium, today's level is 80 mg new + $80(0.5)$ from yesterday + $80(0.5)^2$ from the day before + $\cdots$, an infinite geometric series with $a_1 = 80$ and $r = 0.5$. $S = \dfrac{80}{1 - 0.5} = \dfrac{80}{0.5} = 160$ mg. (Reality check: equilibrium should be larger than a single dose – 160 mg, double of 80, makes sense because with 50% retention the long-run level settles at twice the daily intake.)