Fundamental Theorem of Algebra

Name: ____________	Date: ____________	Score: ______ / 32

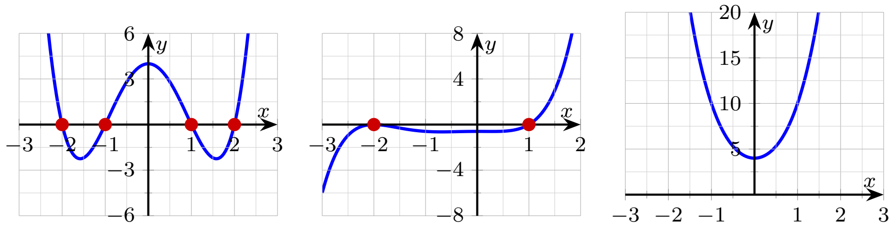
Quick Review

The **Fundamental Theorem of Algebra** (FTA) is the capstone result of polynomial theory: every polynomial of degree $n \geq 1$ with complex coefficients has exactly n roots in $\mathbb{C}$, counted with multiplicity. So a degree-3 polynomial has 3 roots, a degree-7 has 7, and so on — no exceptions, once you allow complex roots.

Multiplicity is the count of how many times a root appears in the factorization. The polynomial $p(x) = (x-2)^3(x+1)$ is degree 4 and has roots $2, 2, 2, -1$ — still four roots once 2 is counted three times.

Complex Conjugate Root Theorem (real coefficients only): if a polynomial has *real* coefficients and $a + bi$ is a root with $b \neq 0$, then $a - bi$ is automatically a root too. Non-real roots come in conjugate pairs. So a polynomial with real coefficients always has an even number of non-real roots, and an odd-degree real-coefficient polynomial *must* have at least one real root.

Graphically: each real root is an x-intercept (a crossing if odd-multiplicity, a tangent touch if even-multiplicity), and each complex conjugate pair contributes *no* x-intercept. The three graphs below show how a degree-4 polynomial's x-intercepts add up to 4 once you count complex roots.



Left: 4 real roots. Middle: 1 root of multiplicity 2 plus 1 root of multiplicity 1 plus a conjugate pair — that's 5 roots total (degree 5). Right: degree 4 with *zero* real roots — all four roots are complex.

PRACTICE

Apply the Fundamental Theorem of Algebra and the Conjugate Root Theorem.

1. State the number of roots (counted with multiplicity, in $\mathbb{C}$) of any degree-n polynomial with $n \geq 1$. ________
2. How many roots (over $\mathbb{C}$, with multiplicity) does $f(x) = x^2 + 1$ have? Find them. ________
3. A cubic polynomial has $x = 2$ as a double root and $x = -3$ as a single root. How many roots does it have, counting multiplicity? ________
4. A polynomial with real coefficients has $3 + 2i$ as a root. Name another root that must also be present. ________
5. A polynomial with real coefficients has roots 1, -2, and $4 + i$ (each at least once). What is the smallest possible degree? ________
6. State the degree of $p(x) = (x-1)^3(x+2)^2(x^2+4)$ and the number of complex roots (with multiplicity). ________
7. Find the monic real polynomial of smallest degree with roots $x = 5$ and $x = 1 + 2i$. ________
8. A degree-4 polynomial with real coefficients has roots -2, $1 + 3i$, and $1 - 3i$. Must the remaining root be real, non-real, or could it be either? ________
9. A polynomial with real coefficients, leading coefficient 1, has roots 2 (multiplicity 2), -1, and $4 - i$. Find the smallest possible degree. ________
10. True or false: *Every degree-5 polynomial with real coefficients has exactly five real roots.* ________
11. True or false: *A polynomial of degree ≥ 1 must have at least one root in $\mathbb{C}$.* ________
12. A degree-3 polynomial with real coefficients has exactly one real x-intercept. What can you say about the other two roots? ________

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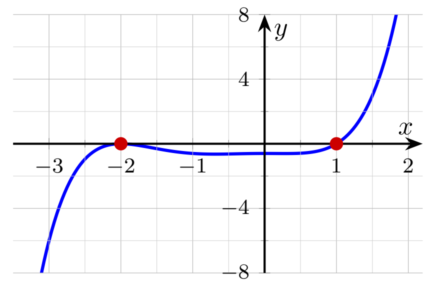
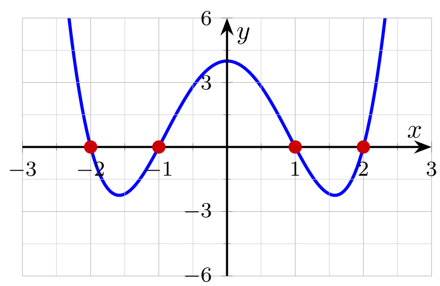
13. The graph of a degree-5 polynomial $r(x)$ with real coefficients touches the x-axis at $x = -2$ and crosses it at $x = 1$, with no other x-intercepts. What can you conclude about the remaining zeros? ________



14. How many complex roots (with multiplicity) does $p(x) = x^4 - 5x^2 + 4$ have? Show that they are all real. ________



15. Find all complex roots of $x^3 - 8 = 0$ (cubic, real coefficients). Hint: factor as a difference of cubes. ________

16. How many real roots can a degree-6 polynomial with real coefficients have? List all possibilities. ________

17. How many real roots must a degree-7 polynomial with real coefficients have at least? ________

18. The polynomial $p(x) = x^2 + 2x + 5$ has real coefficients. Find both roots and verify they form a conjugate pair. ________

19. Build the monic polynomial of smallest degree with real coefficients whose only roots are $2i$ and $-2i$. Write it in standard form. ________

20. True or false: *If a polynomial has complex coefficients (not just real), then its non-real roots must still come in conjugate pairs.* ________

♦ Word Problems

21. A degree-5 polynomial $p(x)$ with real coefficients has roots $x = 3$ (multiplicity 2), $x = -1$, and $x = 2 + i$. List all five roots, counted with multiplicity. ____

22. An engineer's transfer function is a polynomial $p(s)$ of degree 4 with real coefficients. It has a root at $s = -2 + 3i$ and another root at $s = -1$. How many additional roots must $p(s)$ have, and what does the Conjugate Root Theorem say about them? ____

23. Build the monic real polynomial of smallest possible degree whose roots include 1 and i. Then write it as a sum of monomials. ____

24. A polynomial $q(x)$ has degree 6 and real coefficients. Its graph crosses the x-axis exactly twice. According to the FTA and the Conjugate Root Theorem, how many non-real complex roots must $q(x)$ have? ____

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Additional Practice

25. A degree-8 polynomial has how many complex roots, counting multiplicity? ________

26. A real polynomial has root $5-2i$. Name the required conjugate root. ________

27. The roots of a polynomial are -3, 4, and 4 again. What is its least degree? ________

28. How many roots does $(x-1)^2(x+6)^3$ have, counting multiplicity? ________

29. List the roots of $(x+2)(x^2+9)=0$ over the complex numbers. ________

30. A real degree-5 polynomial has roots 1, $2+i$, and -4. What root is still forced? ________

31. Can a real degree-4 polynomial have exactly 3 non-real roots? ________

32. If $x=7$ is a triple root, how many times does it count toward the degree? ________

Answer Keys

1. n
2. 2 roots: $x = \pm i$
3. 3
4. $3 - 2i$
5. 4
6. degree 7, 7 roots
7. $x^3 - 7x^2 + 15x - 25$
8. must be real
9. 5
10. false
11. true
12. a complex conjugate pair
13. two complex roots, a conjugate pair
14. 4 roots: $\pm 1, \pm 2$
15. $x = 2, \ -1 \pm i\sqrt{3}$
16. $0, \ 2, \ 4,$ or 6
17. at least 1
18. $x = -1 \pm 2i$
19. $x^2 + 4$
20. false
21. $3, \ 3, \ -1, \ 2+i, \ 2-i$
22. 2 more; one is $-2 - 3i$
23. $x^3 - x^2 + x - 1$
24. 4 non-real roots (two conjugate pairs)

Additional Practice Answers

25. 8
26. $5 + 2i$
27. 3
28. 5
29. $-2, \ \pm 3i$
30. $2 - i$
31. no
32. 3

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. That's the FTA in one line: degree-n polynomial $\Rightarrow$ exactly n complex roots, counting multiplicity. Real or complex coefficients — doesn't matter for the count.

2. Degree 2, so 2 roots by the FTA. Solve: $x^2 = -1$, so $x = \pm i$. (This example is the reason complex numbers exist — without them the FTA fails.)

3. One steady path is: $x = 2$ counts twice (multiplicity 2), $x = -3$ counts once. Total: $2 + 1 = 3$. The FTA matches: degree $3 \Rightarrow 3$ roots. That gives a quick check on the answer.

4. Conjugate Root Theorem: with real coefficients, non-real roots come in conjugate pairs. The conjugate of $3 + 2i$ is $3 - 2i$ (flip only the imaginary sign).

5. The non-real root $4 + i$ drags its conjugate $4 - i$ along. Required roots: $1, -2, 4+i, 4-i$ — four distinct roots, so degree at least 4.

6. Add exponents: $3 + 2 + 2 = 7$. By the FTA the count of complex roots equals the degree, so 7 complex roots. (The factor $x^2 + 4$ supplies the conjugate pair $\pm 2i$.)

7. Real coefficients force the conjugate $1 - 2i$ to be a root. The conjugate pair gives $(x - (1+2i))(x - (1-2i)) = (x-1)^2 + 4 = x^2 - 2x + 5$. Multiply by $(x - 5)$: $(x-5)(x^2 - 2x + 5) = x^3 - 7x^2 + 15x - 25$.

8. Three roots given; one slot left. A non-real fourth root would force its conjugate too, requiring two more slots — but only one is available. So the fourth root must be real.

9. Required roots with multiplicity: $2, 2, -1, 4 - i$. Conjugate forces $4 + i$. Five required roots $\Rightarrow$ degree at least 5. The polynomial is $(x-2)^2(x+1)(x-(4-i))(x-(4+i))$.

10. Five *complex* roots — yes (FTA). Five *real* roots — not required. Quick check: $p(x) = (x-1)(x^2+1)(x^2+4)$ has degree 5 and only one real root ($x = 1$); the other four are $\pm i, \pm 2i$.

11. This is the existence form of the FTA — and historically the original statement (Gauss, 1799). Every non-constant complex polynomial has at least one complex root.

12. Three complex roots in total. One is real, leaving two unaccounted for. Real coefficients + non-real root $\Rightarrow$ the other is its conjugate. So roots are r, $a + bi$, $a - bi$ for some real r, a, b.

13. A touch is an even-multiplicity root — here multiplicity 2 at $x = -2$. A crossing is odd-multiplicity — here multiplicity 1 at $x = 1$. That accounts for $2 + 1 = 3$ roots. Two remain. Since the graph shows no more x-intercepts, those two are non-real, and real coefficients force them into a conjugate pair.

14. Degree 4, so 4 complex roots by the FTA. Factor: $p(x) = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$, giving roots $\pm 1, \pm 2$, all real. The graph confirms: four x-intercepts.

15. Difference of cubes: $x^3 - 8 = (x-2)(x^2+2x+4)$. The factor $x - 2$ gives $x = 2$. The quadratic factor $x^2 + 2x + 4$ has discriminant $4 - 16 = -12 < 0$, so its roots are $x = \dfrac{-2 \pm \sqrt{-12}}{2} = -1 \pm i\sqrt{3}$ — a conjugate pair, as the theorem requires.

16. Total roots: 6 (FTA). Non-real roots come in conjugate pairs, so the count of non-real roots is even: $0, 2, 4,$ or 6. The count of real roots is whatever is left: $6, 4, 2,$ or 0. (An even-degree polynomial doesn't have to have a real root — $x^6 + 1$ has none.)

17. A careful way to see it: 7 total roots; non-real roots come in pairs, so an even number of them are non-real. An even number plus a real-root count equals 7 (odd), so the real-root count is odd — which means at least 1. This is also the Intermediate-Value-Theorem argument: an odd-degree real polynomial goes from $-\infty$ to $+\infty$ and must cross zero somewhere. That gives a quick check on the answer.

18. Quadratic formula: $x = \dfrac{-2 \pm \sqrt{4 - 20}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$. The two roots are $-1 + 2i$ and $-1 - 2i$, which differ only in the sign of the imaginary part — a conjugate pair, exactly as the theorem predicts.

19. The two roots are already a conjugate pair, so degree 2 suffices: $(x - 2i)(x + 2i) = x^2 - (2i)^2 = x^2 - (-4) = x^2 + 4$.

20. The Conjugate Root Theorem requires *real* coefficients. With complex coefficients, conjugates need not pair up. Quick check: $p(x) = x - i$ has the single root i; its conjugate $-i$ is not a root.

21. Multiplicity counts: 3 twice, -1 once, $2 + i$ once. That's 4 roots, but the polynomial is degree 5. The Conjugate Root Theorem supplies the fifth: $2 - i$. So the full list (with multiplicity) is $3, 3, -1, 2+i, 2-i$.

22. Degree $4 \Rightarrow 4$ total roots. Two are given ($-2 + 3i$ and -1), so 2 more remain. The Conjugate Root Theorem forces $-2 - 3i$ to be a root (the conjugate of $-2 + 3i$). The last root could be any real number, or another conjugate pair split across the remaining slot — but since only one slot is left after $-2 - 3i$ takes one, that final root must be real.

23. Real coefficients force $-i$ to join i as a root. So required roots are $1, i, -i$, giving degree 3. Multiply: $(x-1)(x-i)(x+i) = (x-1)(x^2+1) = x^3 + x - x^2 - 1 = x^3 - x^2 + x - 1$.

24. Total roots: 6. The graph crosses the x-axis twice, so 2 real roots are visible. (Touches without crossing would also count as real, but the problem says two crossings and no other intercepts.) That leaves $6 - 2 = 4$ non-real roots, which must split into 2 conjugate pairs because the coefficients are real.

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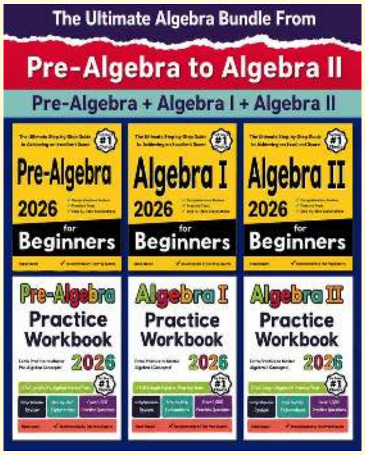

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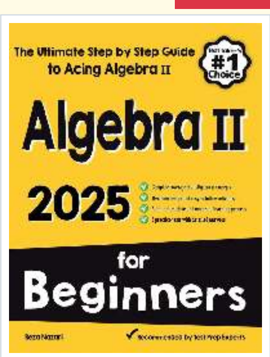

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