Finding Zeros of Polynomials

Name: ______________________	Date: ______________	Score: _____ / 28

Quick Review

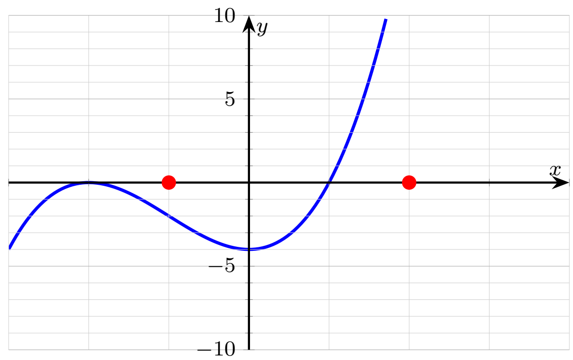
A **zero** of a polynomial $f(x)$ is a value of x that makes $f(x) = 0$. Graphically, zeros are the x-intercepts.

If f is already factored, use the **zero-product property**: a product is zero exactly when at least one factor is zero. So $(x-3)(x+5) = 0$ gives $x = 3$ or $x = -5$.

For a quadratic in standard form, factor and apply the zero-product property: $x^2 - 5x + 6 = (x-2)(x-3) = 0$ has zeros $x = 2, 3$.

For cubics and higher, use the **Rational Root Theorem**: any rational zero of $f(x)$ has the form $\pm\frac{p}{q}$ where p | constant term and q | leading coefficient. Test each candidate by plugging in. Once you find a zero c, divide out $(x-c)$ (synthetic division) and factor the resulting depressed polynomial.

Multiplicity: if a factor $(x-c)$ appears with exponent k, then c is a zero of multiplicity k. Graphically: multiplicity 1 means the graph crosses; multiplicity 2 means it touches and bounces back (doesn't cross); multiplicity ≥ 3 flattens through. The graph of $f(x) = (x-2)^2(x+1)$ below illustrates: $x = 2$ is a double zero (touches), $x = -1$ is a simple zero (crosses).



The **Fundamental Theorem of Algebra** says a degree-n polynomial has exactly n complex zeros (counted with multiplicity). Graphs show only the real ones — complex zeros come in conjugate pairs $a \pm bi$ and don't appear as x-intercepts. So a degree-4 polynomial can have 0, 2, or 4 real zeros (and the rest are complex).

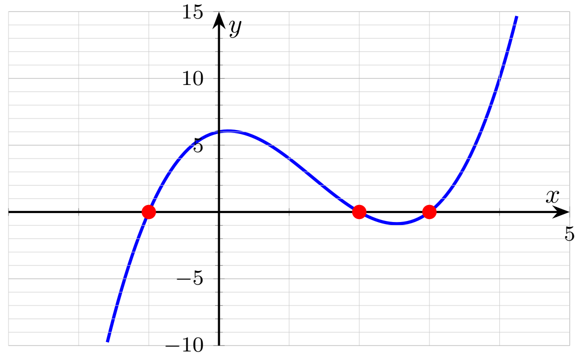
PRACTICE

Find all real zeros of each polynomial, including multiplicities when stated.

1. Zeros of $f(x) = (x-3)(x+5)$ ________
2. Zeros of $g(x) = x^2 - 5x + 6$ ________
3. Rational root candidates for $2x^3 + 3x^2 - 8x + 3$ ________
4. Find a zero of $f(x) = x^3 - 3x^2 + 2$ ________
5. Given that $x = 2$ is a zero of $f(x) = x^3 - 6x^2 + 11x - 6$, find the other zeros ________
6. Zeros of $f(x) = (x-2)^2(x+1)$, with multiplicity ________

Scan Me

7. All real zeros of $f(x) = x^3 - 4x^2 + x + 6$ from its graph below: ________

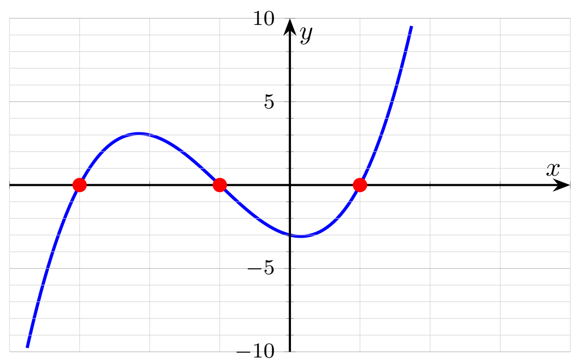


8. If $x = -2$ is a zero of $p(x) = x^3 + kx^2 - 4x - 12$, find k ________

9. Real zeros of $f(x) = x^3 - 2x^2 + 4x - 8$ ________

10. Zeros of $f(x) = x^2(x-3)(x+2)$ ________

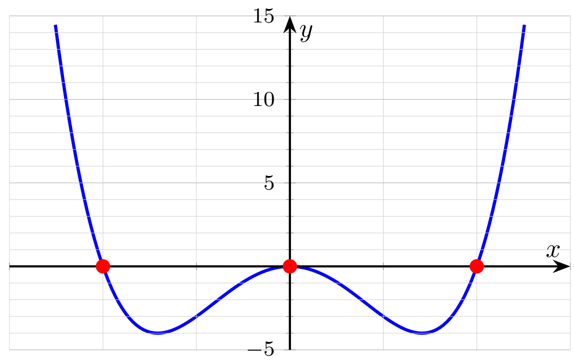
11. The graph shows a cubic with three distinct real zeros. State its zeros: ________



12. Real zero of $f(x) = x^3 + 8$ ________

13. Number of real zeros of $f(x) = x^4 + 1$ ________

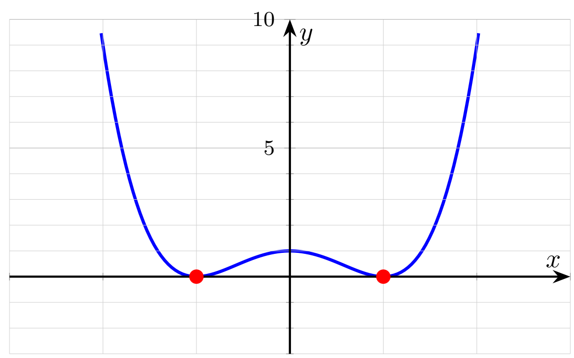
14. The graph below has x-intercepts; list all real zeros: ________



15. Find all zeros of $f(x) = x^3 + x^2 - 4x - 4$ ________

16. Identify the multiplicity of $x = 4 in f(x) = (x-4)^3(x+1)$ ________

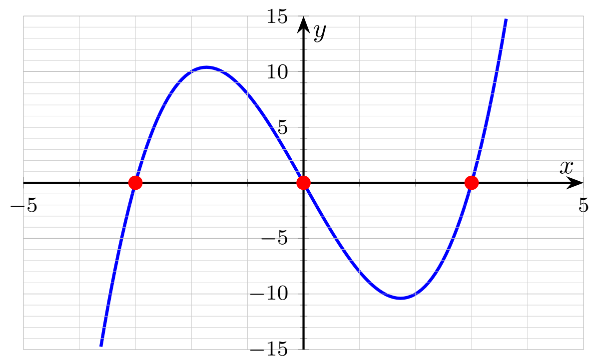
17. The graph below shows a polynomial with a double zero. Identify the zeros and their multiplicities: ________



18. Zeros of $f(x) = 2(x-1)(x+3)(x-5)$ ________

19. Use the Fundamental Theorem to state how many complex zeros (with multiplicity) $x^5 - 7x^2 + 1$ has ________

20. Real zeros of the polynomial shown below: ________



Word Problems

21. A polynomial $p(x) = x^3 + x^2 - 9x - 9$ models the net profit (in thousands) of a startup over time. Find all real values of x where the profit is zero. ______

22. A box's volume polynomial is $V(x) = x^3 - 6x^2 + 11x - 6$, and $V(x) = 0$ corresponds to a degenerate box. Find the values of x where the box collapses. ______

23. A polynomial $p(x)$ has zeros at $x = -2, 1$ (double root), and 4, with leading coefficient 1. Write $p(x)$ in factored form and expand to standard form. ______

24. For the polynomial $p(x) = x^4 - 5x^2 + 4$, find all real zeros. ______

Additional Practice

25. Write $3x - 5 + x^3$ in standard form. ________

26. Find the degree of $7x^4 - 2x^2 + 9$. ________

27. Add $(2x^2 + 3x - 1) + (x^2 - 5x + 4)$. ________

28. Subtract $(5x^2 - x + 6) - (2x^2 + 3x - 1)$. ________

Answer Keys

1.	$x=3,\ x=-5$	13.	0
2.	$x=2,\ x=3$	14.	$x=-2,\ x=0,\ x=2$
3.	$\pm1,\pm3,\pm\frac{1}{2},\pm\frac{3}{2}$	15.	$x=-1,\ x=2,\ x=-2$
4.	$x=1$	16.	3
5.	$x=1,\ x=3$	17.	$x=-1,1\ (each multiplicity 2)$
6.	$x=2(mult.2),\ x=-1(mult.1)$	18.	$x=1,\ x=-3,\ x=5$
7.	$x=-1,\ x=2,\ x=3$	19.	5
8.	$k=3$	20.	$x=-3,\ x=0,\ x=3$
9.	$x=2$	21.	$x=-3,\ x=-1,\ x=3$
10.	$x=0(mult.2),\ x=3,\ x=-2$	22.	$x=1,\ x=2,\ x=3$
11.	$x=-3,\ x=-1,\ x=1$	23.	$p(x)=(x+2)(x-1)^2(x-4)$
12.	$x=-2$	24.	$x=-2,-1,1,2$

Additional Practice Answers

25.	x^3+3x-5	27.	$3x^2-2x+3$
26.	4	28.	$3x^2-4x+7$

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. A careful way to see it: Zero-product: $x-3=0$ or $x+5=0$. This is the part to check before moving on, because it keeps the answer tied to the original question.

2. Factor the quadratic: two numbers with product 6 and sum -5 are $-2,-3$, so $(x-2)(x-3)=0$. By the zero-product property, $x-2=0$ or $x-3=0$, giving $x=2,3$.

3. One steady path is: p divides 3: $\pm1,\pm3$. q divides 2: $\pm1,\pm2$. Candidates: $\pm1,\pm3,\pm\frac{1}{2},\pm\frac{3}{2}$. That gives a quick check on the answer.

4. Test $x=1$: $f(1)=1-3+2=0$ ✓. So $x=1$ is a zero. Other zeros require dividing out $(x-1)$ and solving the resulting quadratic.

5. Since $x=2$ is a zero, divide out $(x-2)$. Synthetic division with $c=2$ and $1,-6,11,-6$ gives bottom row $1,-4,3,0$ (remainder 0 confirms the zero). The quotient x^2-4x+3 factors as $(x-1)(x-3)$, so the other zeros are $x=1$ and $x=3$.

6. Factor $(x-2)$ appears twice, so $x=2$ has multiplicity 2. Factor $(x+1)$ appears once, so $x=-1$ has multiplicity 1. Graphically: $x=2$ touches the axis but doesn't cross; $x=-1$ crosses straight through.

7. Rational candidates $\pm1,\pm2,\pm3,\pm6$. Test $x=-1$: $f(-1)=-1-4-1+6=0$ ✓. Synthetic divide by $c=-1$: quotient $x^2-5x+6=(x-2)(x-3)$. So zeros at $-1,2,3$ — matching the red dots.

8. If $x=-2$ is a zero, then $p(-2)=0$. Substitute: $(-2)^3+k(4)-4(-2)-12=-8+4k+8-12=4k-12$. Set $4k-12=0$, so $k=3$.

9. Group: $x^2(x-2)+4(x-2)=(x-2)(x^2+4)$. The first factor gives $x=2$; the second gives $x^2=-4$, so $x=\pm2i$ (complex, not real). Only real zero: $x=2$.

10. The factor $x=x^1$ appears twice (from x^2), giving $x=0$ with multiplicity 2. The others are simple.

11. Reading the red dots from the graph: $x=-3,-1,1$. (The polynomial is $f(x)=x^3+3x^2-x-3=(x+3)(x+1)(x-1)$.)

12. Sum of cubes: $x^3+8=(x+2)(x^2-2x+4)$. Real zero: $x=-2$. The quadratic factor has discriminant $4-16=-12<0$, so its two zeros are complex.

13. A careful way to see it: $x^4+1>0$ for all real x (sum of a nonneg and a positive). All four zeros are complex. That gives a quick check on the answer.

14. Factor: $f(x)=x^4-4x^2=x^2(x^2-4)=x^2(x-2)(x+2)$. Zeros: $x=0$ (mult. 2), $x=2,-2$ (each mult. 1).

15. One steady path is: Group: $x^2(x+1)-4(x+1)=(x+1)(x^2-4)=(x+1)(x-2)(x+2)$. Zeros: $-1,2,-2$. That gives a quick check on the answer.

16. Start with the key idea: Factor $(x-4)$ appears with exponent 3. This is the part to check before moving on, because it keeps the answer tied to the original question.

17. The polynomial is $f(x)=x^4-2x^2+1=(x^2-1)^2=(x-1)^2(x+1)^2$. Both $x=1$ and $x=-1$ have multiplicity 2 — the graph touches the axis at each but doesn't cross.

18. Zero-product: each factor in turn. The leading coefficient 2 doesn't affect zeros.

19. Degree $n\Rightarrow$ exactly n complex zeros, counted with multiplicity. Real zeros are a subset.

20. The polynomial is $f(x)=x^3-9x=x(x-3)(x+3)$. Real zeros: $-3,0,3$. Notice the odd symmetry of the curve — f is an odd function.

21. Group: $x^2(x+1)-9(x+1)=(x+1)(x^2-9)=(x+1)(x-3)(x+3)$. Real zeros: $x=-3,-1,3$. (In context, only the positive value $x=3$ might be physically meaningful, depending on the model's domain.)

22. Try $x=1$: $V(1)=1-6+11-6=0$ ✓. Synthetic divide by $c=1$: quotient $x^2-5x+6=(x-2)(x-3)$. So zeros at $1,2,3$. All three are positive and physically realizable as degenerate-box dimensions.

23. From the zero list: $(x+2)$, $(x-1)^2$ (double root), $(x-4)$. So $p(x)=(x+2)(x-1)^2(x-4)$. Expanded: first multiply $(x-1)^2=x^2-2x+1$, then $(x+2)(x-4)=x^2-2x-8$. Product of two quadratics: $(x^2-2x+1)(x^2-2x-8)$. Let $u=x^2-2x$ to see the pattern: $(u+1)(u-8)=u^2-7u-8=(x^2-2x)^2-7(x^2-2x)-8=x^4-4x^3+4x^2-7x^2+14x-8=x^4-4x^3-3x^2+14x-8$.

24. Treat as a quadratic in $u=x^2$: $u^2-5u+4=(u-1)(u-4)=0$, so $u=1$ or $u=4$. Then $x^2=1\Rightarrow x=\pm1$; $x^2=4\Rightarrow x=\pm2$. Four real zeros: $-2,-1,1,2$. (All real because both u-values are positive.)

Scan Me