Complex Conjugates and Modulus

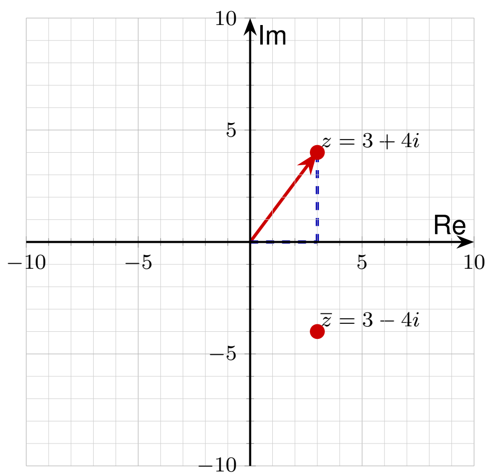
Name: ____________	Date: ____________	Score: ______ / 31

Quick Review

The **conjugate** of $z = a + bi$ is $\overline{z} = a - bi$ — flip the imaginary sign, leave the real part alone. Geometrically, the conjugate is the reflection of z across the real axis. The **modulus** (or absolute value) of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$ — the distance from z to the origin on the Argand plane. Distances are never negative, so a negative modulus is always a mistake.

Three facts you should know cold. First, $z\,\overline{z} = |z|^2$ for every complex z (this is why multiplying by the conjugate clears imaginary parts in division). Second, conjugation distributes over addition and multiplication: $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$. Third, $|zw| = |z|\,|w|$ (moduli *multiply*), but $|z + w| \leq |z| + |w|$ in general (the triangle inequality, with equality only when z and w point the same direction).

The figure shows $z = 3 + 4i$ and its conjugate $\overline{z} = 3 - 4i$ as mirror images across the real axis, with the modulus $|z| = 5$ shown as the hypotenuse of the 3-4-5 triangle.



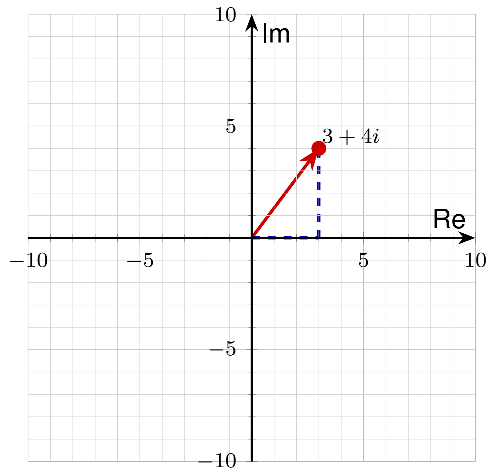
The dashed legs are length 3 (real part) and 4 (imaginary part). The red arrow is the modulus $|z| = \sqrt{3^2 + 4^2} = 5$.

PRACTICE

Find the conjugate or modulus as indicated. Write all answers in standard form.

1. $\overline{4 + 3i}$ ________

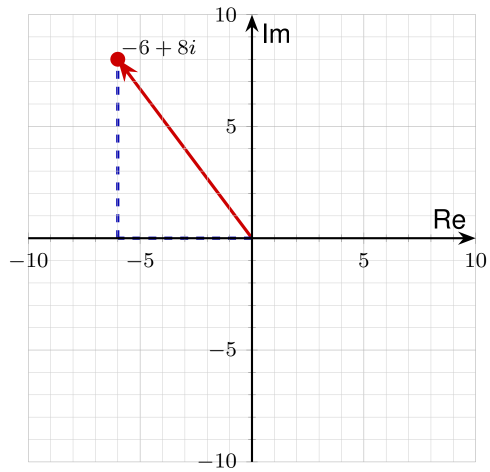
2. $|3+4i|$. Read the modulus as the hypotenuse of the right triangle shown below. ________



3. $\overline{-2-5i}$ ________

4. $z\,\overline{z}$ for $z = 2+3i$ ________

5. $|-6+8i|$. The point sits in the second quadrant; the dashed legs of the right triangle are drawn below. ________



6. $|5-12i|$ ________

7. $\overline{z}$ for $z = \dfrac{1}{2+i}$ ________

8. $\overline{5-12i}$ ________

9. $|z|$ for $z = -3+4i$ ________

10. $\overline{(2+i)+(3-4i)}$ ________

11. $|2-i| \cdot |1+3i|$ ________

12. $|(2-i)(1+3i)|$ ________

13. $\overline{(3+2i)(1-i)}$ ________

14. $|1+i|^4$ ________

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15. $\overline{\overline{z}}$ for $z = 2 - 7i$ ________

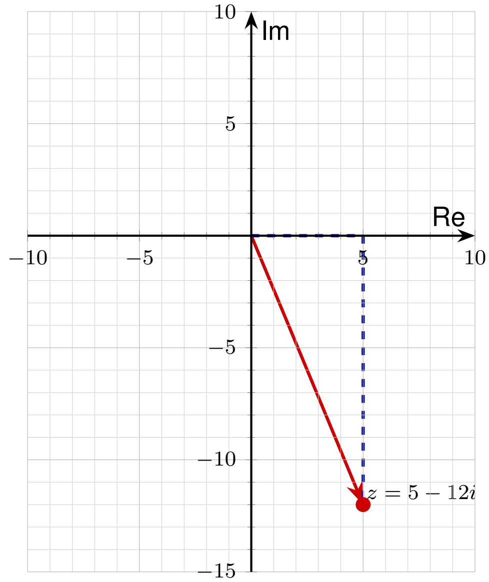
16. Find z if $z\,\overline{z} = 25$, $\text{Re}(z) = 3$, and $\text{Im}(z) > 0$. ________

17. $|3 - 2i|^2$ ________

18. $\overline{i}$ ________

19. $\left|\dfrac{3+4i}{2-i}\right|$ ________

20. On the Argand plane, find the modulus and conjugate of $z = 5 - 12i$. Use the right triangle in the figure. ________



Word Problems

21. In an AC circuit the impedance is $Z = 8 + 6i$ ohms. The magnitude of the impedance is $|Z|$. Find $|Z|$ and explain what it represents physically. ____

22. A point P in the complex plane satisfies $|P - (3+i)| = 5$, meaning P is exactly 5 units from the point $3 + i$. Describe the set of all possible P geometrically. ____

23. Show that for $z = 1 + i$, the identity $|z|^2 = z\,\overline{z}$ holds by computing both sides. ____

24. Two complex numbers have moduli $|z| = 6$ and $|w| = 8$. What is the largest possible value of $|z + w|$, and what is the smallest? (Both z and w are non-zero.) ____

Additional Practice

25. Add $(3 + 2i) + (5 - i)$. ________

26. Subtract $(4 - i) - (1 + 6i)$. ________

27. Multiply $(2 + 3i)(1 - i)$. ________

28. Simplify i^{17}. ________

29. Simplify i^{22}. ________

30. Find the conjugate of $6 - 5i$. ________

31. Find $|3 + 4i|$. ________

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Answer Keys

1. $\boxed{4-3i}$
2. $\boxed{5}$
3. $\boxed{-2+5i}$
4. $\boxed{13}$
5. $\boxed{10}$
6. $\boxed{13}$
7. $\boxed{\dfrac{1}{2-i}}$
8. $\boxed{5+12i}$
9. $\boxed{5}$
10. $\boxed{5+3i}$
11. $\boxed{5\sqrt{2}}$
12. $\boxed{5\sqrt{2}}$
13. $\boxed{5+i}$
14. $\boxed{4}$
15. $\boxed{2-7i}$
16. $\boxed{z=3+4i}$
17. $\boxed{13}$
18. $\boxed{-i}$
19. $\boxed{\sqrt{5}}$
20. $\boxed{|z|=13,\ \overline{z}=5+12i}$
21. $\boxed{|Z|=10 \text{ ohms}}$
22. $\boxed{\text{a circle of radius } 5 \text{ centered at } 3+i}$
23. $\boxed{\text{both equal } 2}$
24. $\boxed{\text{largest } 14,\ \text{smallest } 2}$

Additional Practice Answers

25. $\boxed{8+i}$
26. $\boxed{3-7i}$
27. $\boxed{5+i}$
28. $\boxed{i}$
29. $\boxed{-1}$
30. $\boxed{6+5i}$
31. $\boxed{5}$

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. Taking a conjugate flips only the imaginary sign and leaves the real part untouched: $4+3i \to 4-3i$. Geometrically this reflects the point across the real axis.

2. Keep the rule visible: $\sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$. The 3-4-5 Pythagorean triple makes this come out clean — the dashed legs are 3 and 4, the arrow is the hypotenuse. That gives a quick check on the answer.

3. Real part stays -2; imaginary part flips from -5 to $+5$. Negatives on the real side don't change.

4. Start with the key idea: $(2+3i)(2-3i) = 4+9 = 13$. This equals $|z|^2 = (\sqrt{13})^2 = 13$, confirming $z\overline{z} = |z|^2$. That gives a quick check on the answer.

5. A careful way to see it: $\sqrt{36+64} = \sqrt{100} = 10$. Negative real part doesn't change the modulus — you square it. The triangle has legs 6 and 8, giving hypotenuse 10. That gives a quick check on the answer.

6. Square each part and add under the root: $\sqrt{5^2+(-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$. The negative sign disappears when you square, and the 5-12-13 triple gives a clean answer.

7. Conjugation distributes over division: $\overline{1/(2+i)} = \overline{1}/\overline{2+i} = 1/(2-i)$. (You can rationalize this further to $\dfrac{2+i}{5}$, but $\dfrac{1}{2-i}$ is already a valid expression for $\overline{z}$.)

8. Flip only the imaginary sign — the $-12i$ becomes $+12i$ while the real 5 stays put: $5+12i$.

9. A careful way to see it: $\sqrt{9+16} = 5$. Same 3-4-5 triple, just reflected to the second quadrant. Modulus is unchanged. That gives a quick check on the answer.

10. Add first: $(2+i)+(3-4i) = 5-3i$. Now take the conjugate: $\overline{5-3i} = 5+3i$. (Or use the identity $\overline{z+w} = \overline{z}+\overline{w}$: $\overline{2+i}+\overline{3-4i} = (2-i)+(3+4i) = 5+3i$.)

11. One steady path is: $|2-i| = \sqrt{5}$ and $|1+3i| = \sqrt{10}$. Multiply: $\sqrt{5}\cdot\sqrt{10} = \sqrt{50} = 5\sqrt{2}$. (Or use $|zw| = |z||w|$ and compute $|(2-i)(1+3i)|$ directly.) That gives a quick check on the answer.

12. By $|zw| = |z||w|$ this equals $|2-i|\cdot|1+3i| = \sqrt{5}\cdot\sqrt{10} = 5\sqrt{2}$. (Sanity check: FOIL $(2-i)(1+3i) = 2+6i-i-3i^2 = 5+5i$, then $|5+5i| = \sqrt{50} = 5\sqrt{2}$ ✓.)

13. Use $\overline{zw} = \overline{z}\,\overline{w}$: $\overline{3+2i}\cdot\overline{1-i} = (3-2i)(1+i) = 3+3i-2i-2i^2 = 3+i+2 = 5+i$. (Or multiply first: $(3+2i)(1-i) = 3-3i+2i-2i^2 = 5-i$; conjugate: $5+i$.)

14. Keep the rule visible: $|1+i| = \sqrt{2}$, so $|1+i|^4 = (\sqrt{2})^4 = 4$. (Alternative: $|1+i|^4 = |(1+i)^4| = |-4| = 4$ using $(1+i)^4 = -4$ from Section 4.2.) That gives a quick check on the answer.

15. Conjugation twice undoes itself: $\overline{\overline{2-7i}} = \overline{2+7i} = 2-7i = z$. The conjugate of the conjugate is always the original.

16. Start with the key idea: $z\overline{z} = |z|^2 = 25$, so $|z| = 5$. With real part 3, the imaginary part satisfies $3^2+b^2 = 25$, giving $b^2 = 16$, so $b = \pm 4$. The constraint $\mathrm{Im}(z) > 0$ picks $b = 4$. So $z = 3+4i$. That gives a quick check on the answer.

17. A careful way to see it: $|3-2i|^2 = 3^2+(-2)^2 = 9+4 = 13$. (Or use $|z|^2 = z\overline{z} = (3-2i)(3+2i) = 13$.) That gives a quick check on the answer.

18. Write i in full form as $0+1i$. Flipping the imaginary sign gives $0-1i = -i$. So the conjugate of i is $-i$.

19. Modulus distributes over division: $\left|\dfrac{3+4i}{2-i}\right| = \dfrac{|3+4i|}{|2-i|} = \dfrac{5}{\sqrt{5}} = \sqrt{5}$. (Much faster than rationalizing first.)

20. Modulus: $|z| = \sqrt{5^2+(-12)^2} = \sqrt{25+144} = \sqrt{169} = 13$. That's the hypotenuse of the right triangle in the figure. Conjugate: flip the imaginary sign, giving $5+12i$ (reflection across the real axis).

21. A careful way to see it: $|Z| = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$ ohms. Physically, $|Z|$ is the total *apparent* impedance — the ratio of peak voltage to peak current. The real and imaginary parts split that into resistive and reactive components, and the modulus combines them via Pythagoras. That gives a quick check on the answer.

22. Keep the rule visible: $|P-(3+i)|$ is the distance from P to $3+i$ on the Argand plane (think of P and $3+i$ as points). Setting that distance equal to 5 traces out a circle of radius 5 centered at $3+i$ — the complex-number version of $(x-3)^2+(y-1)^2 = 25$. That gives a quick check on the answer.

23. Left side: $|z| = \sqrt{1+1} = \sqrt{2}$, so $|z|^2 = 2$. Right side: $z\overline{z} = (1+i)(1-i) = 1-i^2 = 1+1 = 2$. Equal — exactly as the identity promises. This pattern is why $(c+di)(c-di)$ shows up so often when you rationalize complex denominators.

24. By the triangle inequality, $|z+w| \le |z|+|w| = 14$, with equality when z and w point the same direction. For the smallest, $|z+w| \ge ||z|-|w|| = |6-8| = 2$, with equality when they point opposite directions. So the range is $2 \le |z+w| \le 14$.

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