

Augmented Matrices and Gaussian Elimination

Name: _____ Date: _____ Score: _____ / 29

Q Quick Review

An **augmented matrix** packages a linear system into one rectangular array: coefficients on the left of a vertical bar, constants on the right. The system $\begin{cases} 2x + 3y = 7 \\ x - 4y = -5 \end{cases}$ becomes $\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -4 & -5 \end{array} \right]$. Each row is one equation; the bar separates variables from constants.

Three elementary row operations: (1) *swap* two rows; (2) *scale* a row by a nonzero constant; (3) *add* a multiple of one row to another. All three preserve the solution set. Multiplying a row by 0 is *not* allowed — it destroys information and isn't reversible.

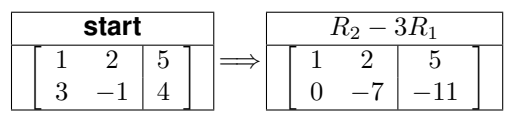
Goal: row-echelon form (REF) — leading 1s stepping down to the right, zeros below each leading entry. Further reduction gives **reduced row-echelon form (RREF)** — zeros above each leading 1 as well. From RREF you can read the solution directly.

Three outcomes from reduction. (1) A row $[0 \ 0 \ \dots \ 0 \ | \ k]$ with $k \neq 0$ represents $0 = k$, a contradiction \Rightarrow *no solution*. (2) A row of all zeros (with 0 on the right) is a redundant equation; combined with a free column, this signals *infinitely many solutions*. (3) Every variable column has a pivot \Rightarrow *unique solution*, read off the right column. **Watch out:** a row of all zeros (including the right side) is *not* a contradiction — it's just a redundant equation. Only the form $0 = k$ with $k \neq 0$ means *no solution*.

PRACTICE

Use elementary row operations to row-reduce. Read off the solution or state its type.

- Write the augmented matrix for $\begin{cases} 2x + 3y = 7 \\ x - 4y = -5 \end{cases}$. _____
- Which of the following is *not* a valid elementary row operation: swap two rows; multiply a row by 0; multiply a row by -3 ; add 2 times row 1 to row 2? _____
- Apply $R_2 \rightarrow R_2 - 3R_1$ to $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 4 \end{array} \right]$. _____



- The RREF of a 2×2 system's augmented matrix is $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array} \right]$. What is the solution? _____

RREF		
1	0	4
0	1	-3

- What does the row $[\ 0 \ 0 \ | \ 5 \]$ tell you about the original system? _____



6. Solve by row reduction: $\begin{cases} x + 2y = 5 \\ 2x + 5y = 11 \end{cases}$.

start		
1	2	5
2	5	11

 \Rightarrow

$R_2 - 2R_1$		
1	2	5
0	1	1

$R_1 - 2R_2$		
1	0	3
0	1	1

7. Solve: $\begin{cases} x + y - z = 4 \\ 2x - y + z = -1 \\ x + 2y + z = 4 \end{cases}$. (Show the row-reduction steps.)

start			
1	1	-1	4
2	-1	1	-1
1	2	1	4

 \Rightarrow

$R_2 - 2R_1, R_3 - R_1$			
1	1	-1	4
0	-3	3	-9
0	1	2	0

$-\frac{1}{3}R_2$			
1	1	-1	4
0	1	-1	3
0	1	2	0

 \Rightarrow

$R_3 - R_2$			
1	1	-1	4
0	1	-1	3
0	0	3	-3

$\frac{1}{3}R_3$			
1	1	-1	4
0	1	-1	3
0	0	1	-1

8. Classify the solution set from this RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

9. Solve: $\begin{cases} 2x + y = 9 \\ 4x - 3y = 7 \end{cases}$.

start		
2	1	9
4	-3	7

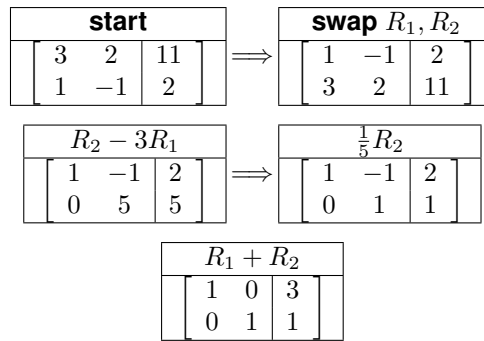
 \Rightarrow

$R_2 - 2R_1$		
2	1	9
0	-5	-11

$-\frac{1}{5}R_2$		
2	1	9
0	1	$\frac{11}{5}$

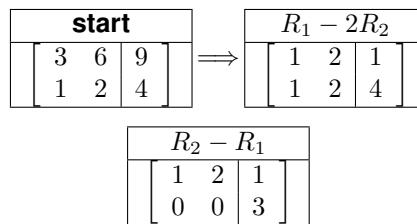


10. Solve: $\begin{cases} 3x + 2y = 11 \\ x - y = 2 \end{cases}$ by row-reduction.

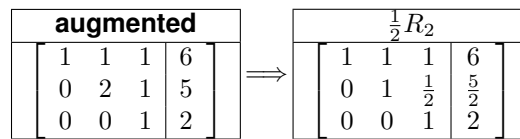


11. Identify the form of this matrix and classify the solution set: $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 5 \end{array} \right]$.

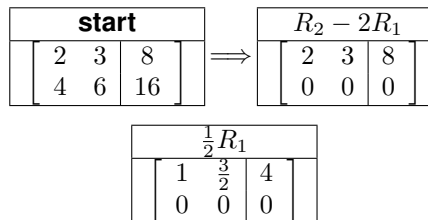
12. Apply $R_1 \rightarrow R_1 - 2R_2$ to $\left[\begin{array}{cc|c} 3 & 6 & 9 \\ 1 & 2 & 4 \end{array} \right]$. What happens?



13. Solve: $\begin{cases} x + y + z = 6 \\ 2y + z = 5 \\ z = 2 \end{cases}$. (Already in echelon-like form — finish with back-substitution.)



14. The augmented matrix $\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 4 & 6 & 16 \end{array} \right]$ row-reduces. What is the solution set?



15. True or false: a row of all zeros in an augmented matrix always signals *no solution*.



16. Solve $\begin{cases} x + 2y - z = 3 \\ 2x + y + z = 4 \\ 3x - y + 2z = 5 \end{cases}$ by row reduction.

augmented				⇒	$R_2 - 2R_1, R_3 - 3R_1$			
1	2	-1	3		1	2	-1	3
2	1	1	4		0	-3	3	-2
3	-1	2	5		0	-7	5	-4

$-\frac{1}{3}R_2$				⇒	$R_3 + 7R_2$			
1	2	-1	3		1	2	-1	3
0	1	-1	$\frac{2}{3}$		0	1	-1	$\frac{2}{3}$
0	-7	5	-4		0	0	-2	$\frac{2}{3}$

17. What does each elementary row operation correspond to, in terms of the original equations? _____

18. Solve $\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$ by row-reduction. _____

start			⇒	$R_2 - 2R_1$		
1	1	3		1	1	3
2	2	6		0	0	0

19. Solve $\begin{cases} x + y + z = 2 \\ x + y + z = 5 \end{cases}$ by row-reduction. Classify. _____

start				⇒	$R_2 - R_1$			
1	1	1	2		1	1	1	2
1	1	1	5		0	0	0	3

20. Solve $\begin{cases} x + y = 5 \\ 2x - 3y = 0 \end{cases}$ by row-reduction. _____

start			⇒	$R_2 - 2R_1$		
1	1	5		1	1	5
2	-3	0		0	-5	-10

$-\frac{1}{5}R_2$			⇒	$R_1 - R_2$		
1	1	5		1	0	3
0	1	2		0	1	2



◆ Word Problems

21. A small business borrows from three banks at different rates. The total borrowed is \$24,000. Bank A's rate is 5%, Bank B's is 6%, and Bank C's is 7%; the total interest for the year is \$1,480. Bank C lent twice as much as Bank A. Use Gaussian elimination on the system to find how much each bank lent. _____
22. A school cafeteria sells three lunches priced at \$4, \$5, and \$6. On Tuesday they sold 90 lunches for \$445 in revenue. The number of \$5 lunches equaled the combined number of \$4 and \$6 lunches. How many of each type sold? _____
23. A delivery service routes packages through three sorting centers. Center A handles 40% of the packages, Center B handles 35%, and Center C handles the rest. The total daily volume is 1,200 packages. Use a matrix equation or Gaussian elimination to find how many packages each center processes. _____
24. A small business builds three product types A, B, and C, each needing time on two machines. One unit of A needs 1 hr on M1 and 2 hrs on M2; one unit of B needs 2 hrs on M1 and 1 hr on M2; one unit of C needs 1 hr on M1 and 1 hr on M2. M1 has 12 hours available and M2 has 14 hours. The company also wants to make 9 units total. Use Gaussian elimination to find the number of each. _____

Additional Practice

25. State the dimensions of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. _____
26. Add $\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix}$. _____
27. Subtract $\begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$. _____
28. Find $\det \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$. _____
29. Find entry a_{21} in $\begin{bmatrix} 8 & 9 \\ -3 & 4 \end{bmatrix}$. _____



Answer Keys

<p>1. $\left[\begin{array}{cc c} 2 & 3 & 7 \\ 1 & -4 & -5 \end{array} \right]$</p> <p>2. multiply a row by 0</p> <p>3. $\left[\begin{array}{cc c} 1 & 2 & 5 \\ 0 & -7 & -11 \end{array} \right]$</p> <p>4. $(x, y) = (4, -3)$</p> <p>5. no solution (inconsistent)</p> <p>6. $(x, y) = (3, 1)$</p> <p>7. $(x, y, z) = (1, 2, -1)$</p> <p>8. infinitely many solutions</p> <p>9. $(x, y) = \left(\frac{17}{5}, \frac{11}{5}\right)$</p> <p>10. $(x, y) = (3, 1)$</p> <p>11. no solution</p> <p>Additional Practice Answers</p> <p>25. 2×3</p> <p>26. $\left[\begin{array}{cc} 4 & 3 \\ 7 & 6 \end{array} \right]$</p>	<p>12. inconsistent: no solution</p> <p>13. $(x, y, z) = \left(\frac{5}{2}, \frac{3}{2}, 2\right)$</p> <p>14. infinitely many: y free</p> <p>15. false</p> <p>16. $(x, y, z) = \left(2, \frac{1}{3}, -\frac{1}{3}\right)$</p> <p>17. equation-level operations</p> <p>18. infinitely many: y free</p> <p>19. no solution (inconsistent)</p> <p>20. $(x, y) = (3, 2)$</p> <p>21. $A = \\$4000, B = \\$12,000, C = \\$8000$</p> <p>22. \$4: 25, \$5: 45, \$6: 20</p> <p>23. $A = 480, B = 420, C = 300$</p> <p>24. $A = 5, B = 3, C = 1$</p> <p>27. $\left[\begin{array}{cc} 4 & -2 \\ 3 & 4 \end{array} \right]$</p> <p>28. 2</p> <p>29. -3</p>
--	---

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

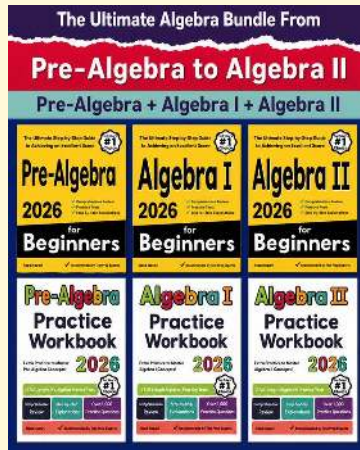
Step-by-Step Explanations

1. Row 1: coefficients 2, 3 and constant 7. Row 2: coefficients 1, -4 and constant -5. The vertical bar marks where coefficients end and constants begin.
2. The three legal operations are swap, scale by a *nonzero* constant, and add a multiple of one row to another. Multiplying by 0 wipes out the row's information and can't be undone — not reversible, so not allowed.
3. Operate on row 2 only: $(3 - 3(1), -1 - 3(2), 4 - 3(5)) = (0, -7, -11)$. Row 1 stays put. (The operation $R_2 \rightarrow R_2 - 3R_1$ zeros out the first column of row 2, the opening move in Gaussian elimination.)
4. In RREF, leading 1s in columns for x and y mean the right column gives the variable values directly: $x = 4, y = -3$.
5. Read the row as an equation: $0x + 0y = 5$, which simplifies to $0 = 5$ — impossible. Whenever row reduction produces $[0 \ 0 \ \dots \ 0 \ | \ k]$ with $k \neq 0$, the system is inconsistent.
6. Start with $[1, 2|5; 2, 5|11]$. $R_2 \rightarrow R_2 - 2R_1$ gives $[1, 2|5; 0, 1|1]$. Then $R_1 \rightarrow R_1 - 2R_2$ gives $[1, 0|3; 0, 1|1]$. RREF reads $x = 3, y = 1$. Verify: $3 + 2 = 5 \checkmark, 6 + 5 = 11 \checkmark$.
7. Back-substitute: row 3 gives $z = -1$. Row 2: $y - z = 3 \Rightarrow y + 1 = 3 \Rightarrow y = 2$. Row 1: $x + y - z = 4 \Rightarrow x + 2 + 1 = 4 \Rightarrow x = 1$. Check: $1 + 2 - (-1) = 4, 2 - 2 + (-1) = -1, 1 + 4 + (-1) = 4$ all \checkmark .
8. The third row $0 = 0$ is redundant, not contradictory. Columns 1 and 2 have pivots, but column 3 (for z) has no pivot — z is free. With a free variable, infinitely many solutions: $x = 5 - 2z, y = 3 + z, z \in \mathbb{R}$.
9. From row 2: $y = \frac{11}{5}$. Row 1: $2x + y = 9 \Rightarrow 2x = 9 - \frac{11}{5} = \frac{34}{5} \Rightarrow x = \frac{17}{5}$. Verify: $2\left(\frac{17}{5}\right) + \frac{11}{5} = \frac{45}{5} = 9 \checkmark; 4\left(\frac{17}{5}\right) - 3\left(\frac{11}{5}\right) = \frac{68-33}{5} = \frac{35}{5} = 7 \checkmark$.
10. Swap rows first so the leading entry is 1 (cleaner pivots). Then clear below the pivot. Final RREF reads $x = 3, y = 1$. Verify: $3(3) + 2(1) = 11 \checkmark$ and $3 - 1 = 2 \checkmark$.
11. Row 2 says $0x + 0y = 5$, which is impossible. Inconsistent system; the two equations describe parallel lines.
12. Apply $R_1 - 2R_2: (3 - 2, 6 - 4, 9 - 8) = (1, 2, 1)$. Then $R_2 - R_1: (0, 0, 3)$. The row $[0, 0|3]$ is the contradiction $0 = 3$, so no solution. (Row reduction reveals the truth: these equations don't agree.)
13. Row 3 gives $z = 2$ directly. Row 2: $2y + z = 5 \Rightarrow 2y + 2 = 5 \Rightarrow y = \frac{3}{2}$. Row 1: $x + y + z = 6 \Rightarrow x + \frac{3}{2} + 2 = 6 \Rightarrow x = \frac{5}{2}$. So $(x, y, z) = \left(\frac{5}{2}, \frac{3}{2}, 2\right)$. Verify: $\frac{5}{2} + \frac{3}{2} + 2 = 6 \checkmark, 2\left(\frac{3}{2}\right) + 2 = 5 \checkmark, z = 2 \checkmark$.
14. Row 2 becomes all zeros — a redundant equation, not a contradiction. The surviving equation is $x + \frac{3}{2}y = 4$, giving the solution set $x = 4 - \frac{3}{2}y$ with $y \in \mathbb{R}$. (Two equations, one line on the plane — infinitely many points.)
15. A row of all zeros (including the constant column) reads $0 = 0$ — always true, a redundant equation. The *no-solution* signal is $[0 \ 0 \ \dots \ 0 \ | \ k]$ with $k \neq 0$ — a

- contradiction. Watch the right side: it makes the difference.
16. Row 3: $-2z = \frac{2}{3} \Rightarrow z = -\frac{1}{3}$. Row 2: $y - z = \frac{2}{3} \Rightarrow y = \frac{2}{3} + \left(-\frac{1}{3}\right) = \frac{1}{3}$. Row 1: $x + 2y - z = 3 \Rightarrow x + \frac{2}{3} - \left(-\frac{1}{3}\right) = 3 \Rightarrow x + 1 = 3 \Rightarrow x = 2$. So $(x, y, z) = \left(2, \frac{1}{3}, -\frac{1}{3}\right)$. Verify the second equation: $2\left(\frac{1}{3}\right) + \left(-\frac{1}{3}\right) = \frac{1}{3} = \frac{2}{3}$ \checkmark .
17. Swapping rows = swapping the order of two equations. Scaling a row by $c \neq 0$ = multiplying that equation through by c . Adding a multiple of one row to another = adding a multiple of one equation to another. All preserve the solution set because they're standard algebra moves.
18. Row 2 becomes all zeros — equation 2 is just $2 \times$ equation 1, so it adds no new information. Solution set: $x = 3 - y$, with y free. Geometrically, the two equations are the same line.
19. Row 2 becomes $[0, 0, 0|3]$ — the contradiction $0 = 3$. The two equations describe parallel planes; they never meet.
20. Three row operations cleaned this up. RREF reads $x = 3, y = 2$. Verify: $3 + 2 = 5 \checkmark, 6 - 6 = 0 \checkmark$.
21. Let a, b, c be the amounts (in thousands of dollars). System: $a + b + c = 24, 0.05a + 0.06b + 0.07c = 1.48$, and $c = 2a$. Substitute $c = 2a$ into the first: $a + b + 2a = 24$, so $3a + b = 24$ and $b = 24 - 3a$. Plug into the interest equation (multiplied by 100): $5a + 6b + 7c = 148$, giving $5a + 6(24 - 3a) + 7(2a) = 148 \Rightarrow 5a + 144 - 18a + 14a = 148 \Rightarrow a = 4$. Then $b = 24 - 12 = 12$ and $c = 8$. Final: Bank A \$4000, Bank B \$12,000, Bank C \$8000. Verify total 24,000 \checkmark , interest $200 + 720 + 560 = 1480 \checkmark$, and $c = 2a \checkmark$.
22. Let x, y, z be the counts at \$4, \$5, \$6. System: $x + y + z = 90, 4x + 5y + 6z = 445$, and $y = x + z$. Substitute $y = x + z$ into the count equation: $2x + 2z = 90 \Rightarrow x + z = 45$, so $y = 45$. Plug $y = 45$ into the revenue equation: $4x + 225 + 6z = 445 \Rightarrow 4x + 6z = 220$. Combined with $x + z = 45$ (times 4 gives $4x + 4z = 180$), subtract: $2z = 40 \Rightarrow z = 20$, so $x = 25$. Counts: \$4 lunches = 25, \$5 lunches = 45, \$6 lunches = 20. Verify: $25 + 45 + 20 = 90 \checkmark$; revenue $100 + 225 + 120 = 445 \checkmark$; $y = 45 = 25 + 20 \checkmark$.
23. Direct percentages: $A = 0.40(1200) = 480, B = 0.35(1200) = 420, C = 0.25(1200) = 300$. (As a matrix equation: with $a + b + c = 1200$ and the fixed proportions, the unique solution is integer. Verify: $480 + 420 + 300 = 1200 \checkmark$.)
24. Let a, b, c be the units of A, B, C. System: $a + 2b + c = 12$ (M1), $2a + b + c = 14$ (M2), $a + b + c = 9$ (total). Subtract equation 3 from equation 1: $b = 3$. Subtract equation 3 from equation 2: $a = 5$. Then from equation 3: $c = 9 - 5 - 3 = 1$. Verify M1: $5 + 6 + 1 = 12 \checkmark, M2: 10 + 3 + 1 = 14 \checkmark$, total $5 + 3 + 1 = 9 \checkmark$. Clean integer answer.



Build Algebra Confidence From Pre-Algebra Through Algebra II



The Complete Algebra Success Bundle

Pre-Algebra, Algebra I, and Algebra II in one clear path

Friendly lessons, focused practice, and full-review support for every stage.



Scan for the Bundle

6 Books
3 Courses
1 Path

Bundle Value: Six coordinated books help students review missing skills, learn new algebra topics, and practice until the steps feel natural.

Complete Course Path

- ✓ Starts with Pre-Algebra foundations
- ✓ Moves smoothly into Algebra I skills
- ✓ Extends learning through Algebra II topics
- ✓ Great for review, tutoring, and summer study

One bundle, one steady path.

Step-by-Step Lessons

- ✓ Plain-English explanations students can follow
- ✓ Worked examples that show every important step
- ✓ Common mistakes called out before they stick
- ✓ Skill-building practice after each lesson
- ✓ Helpful for independent study or class support

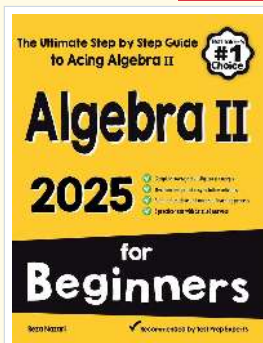
Less guessing. More understanding.

Practice That Sticks

- ✓ Matching practice workbooks for extra repetition
- ✓ Review sets to keep older skills fresh
- ✓ Answer explanations for checking thinking
- ✓ Strong support before tests and final exams
- ✓ Designed to build fluency and confidence

Practice today. Remember tomorrow.

STUDENT FAVORITE • Master Algebra II From the Ground Up



Algebra II for Beginners

Written by a top math teacher & aligned with national and state Algebra II courses. From polynomial functions to logarithms, trigonometry, and rational expressions — explained the easy way.

- ✓ **Complete coverage** of every Algebra II concept — perfect companion to these worksheets
- ✓ **Step-by-step explanations** with worked examples on every topic
- ✓ **QR codes in every chapter** for free video lessons & bonus practice
- ✓ **2 full-length practice tests** with detailed answer keys

- ✓ 100% Guaranteed
- ✓ Lifetime Support
- ✓ Trusted by Teachers

Start Your Algebra Journey Today! →

★ STUDENT'S #1 CHOICE ★

Teacher-recommended • 12,000+ Happy Students

PDF EDITION



Instant download • any device

PAPERBACK



Paperback on Amazon

Hold it in your hands

Pair these free worksheets with *Algebra II for Beginners* and you have a complete self-paced course — concept lessons, daily practice, and full exam-style reviews, all in one path. → EffortlessMath.com/product/algebra-ii-for-beginners