Augmented Matrices and Gaussian Elimination

Name: ________	Date: ________	Score: ____ / 29

Quick Review

An **augmented matrix** packages a linear system into one rectangular array: coefficients on the left of a vertical bar, constants on the right.

The system $\begin{cases} 2x + 3y = 7 \\ x - 4y = -5 \end{cases}$ becomes $\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -4 & -5 \end{array}\right]$. Each row is one equation; the bar separates variables from constants.

Three elementary row operations: (1) *swap* two rows; (2) *scale* a row by a nonzero constant; (3) *add* a multiple of one row to another. All three preserve the solution set. Multiplying a row by 0 is *not* allowed — it destroys information and isn't reversible.

Goal: row-echelon form (REF) — leading 1s stepping down to the right, zeros below each leading entry. Further reduction gives **reduced row-echelon form (RREF)** — zeros above each leading 1 as well. From RREF you can read the solution directly.

Three outcomes from reduction. (1) A row $[0\ 0\ \cdots\ 0 \mid k]$ with $k \neq 0$ represents $0 = k$, a contradiction $\Rightarrow$ *no solution*. (2) A row of all zeros (with 0 on the right) is a redundant equation; combined with a free column, this signals *infinitely many solutions*. (3) Every variable column has a pivot $\Rightarrow$ *unique solution*, read off the right column. **Watch out:** a row of all zeros (including the right side) is *not* a contradiction — it's just a redundant equation. Only the form $0 = k$ with $k \neq 0$ means *no solution*.

PRACTICE

Use elementary row operations to row-reduce. Read off the solution or state its type.

1. Write the augmented matrix for $\begin{cases} 2x + 3y = 7 \\ x - 4y = -5 \end{cases}$. ________

2. Which of the following is *not* a valid elementary row operation: swap two rows; multiply a row by 0; multiply a row by -3; add 2 times row 1 to row 2? ________

3. Apply $R_2 \to R_2 - 3R_1$ to $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 4 \end{array}\right]$. ________

start		$R_2 - 3R_1$
$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & -1 & 4 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -11 \end{array}\right]$

4. The RREF of a 2×2 system's augmented matrix is $\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right]$. What is the solution? ________

RREF
$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -3 \end{array}\right]$

5. What does the row $\left[\begin{array}{cc|c} 0 & 0 & 5 \end{array}\right]$ tell you about the original system? ________

6. Solve by row reduction: $\begin{cases} x+2y=5 \\ 2x+5y=11 \end{cases}$. ________

start		R_2-2R_1
$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 2 & 5 & 11 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc\|c} 1 & 2 & 5 \\ 0 & 1 & 1 \end{array}\right]$

R_1-2R_2
$\left[\begin{array}{cc\|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$

7. Solve: $\begin{cases} x+y-z=4 \\ 2x-y+z=-1 \\ x+2y+z=4 \end{cases}$. (Show the row-reduction steps.) ________

start		$R_2-2R_1,\ R_3-R_1$
$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 4 \\ 2 & -1 & 1 & -1 \\ 1 & 2 & 1 & 4 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 4 \\ 0 & -3 & 3 & -9 \\ 0 & 1 & 2 & 0 \end{array}\right]$

$-\frac{1}{3}R_2$		R_3-R_2
$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & 2 & 0 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 3 & -3 \end{array}\right]$

$\frac{1}{3}R_3$
$\left[\begin{array}{ccc\|c} 1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

8. Classify the solution set from this RREF: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$. ________

9. Solve: $\begin{cases} 2x+y=9 \\ 4x-3y=7 \end{cases}$. ________

start		R_2-2R_1
$\left[\begin{array}{cc\|c} 2 & 1 & 9 \\ 4 & -3 & 7 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc\|c} 2 & 1 & 9 \\ 0 & -5 & -11 \end{array}\right]$

$-\frac{1}{5}R_2$
$\left[\begin{array}{cc\|c} 2 & 1 & 9 \\ 0 & 1 & \frac{11}{5} \end{array}\right]$

10. Solve: $\begin{cases} 3x + 2y = 11 \\ x - y = 2 \end{cases}$ by row-reduction.

start		swap R_1, R_2
$\left[\begin{array}{cc\|c} 3 & 2 & 11 \\ 1 & -1 & 2 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc\|c} 1 & -1 & 2 \\ 3 & 2 & 11 \end{array}\right]$

$R_2 - 3R_1$		$\frac{1}{5}R_2$
$\left[\begin{array}{cc\|c} 1 & -1 & 2 \\ 0 & 5 & 5 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc\|c} 1 & -1 & 2 \\ 0 & 1 & 1 \end{array}\right]$

$R_1 + R_2$
$\left[\begin{array}{cc\|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$

11. Identify the form of this matrix and classify the solution set: $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 5 \end{array}\right]$.

12. Apply $R_1 \to R_1 - 2R_2$ to $\left[\begin{array}{cc|c} 3 & 6 & 9 \\ 1 & 2 & 4 \end{array}\right]$. What happens?

start		$R_1 - 2R_2$
$\left[\begin{array}{cc\|c} 3 & 6 & 9 \\ 1 & 2 & 4 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 1 & 2 & 4 \end{array}\right]$

$R_2 - R_1$
$\left[\begin{array}{cc\|c} 1 & 2 & 1 \\ 0 & 0 & 3 \end{array}\right]$

13. Solve: $\begin{cases} x + y + z = 6 \\ 2y + z = 5 \\ z = 2 \end{cases}$. (Already in echelon-like form — finish with back-substitution.)

augmented		$\frac{1}{2}R_2$
$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 6 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{ccc\|c} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 1 & 2 \end{array}\right]$

14. The augmented matrix $\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 4 & 6 & 16 \end{array}\right]$ row-reduces. What is the solution set?

start		$R_2 - 2R_1$
$\left[\begin{array}{cc\|c} 2 & 3 & 8 \\ 4 & 6 & 16 \end{array}\right]$	$\Longrightarrow$	$\left[\begin{array}{cc\|c} 2 & 3 & 8 \\ 0 & 0 & 0 \end{array}\right]$

$\frac{1}{2}R_1$
$\left[\begin{array}{cc\|c} 1 & \frac{3}{2} & 4 \\ 0 & 0 & 0 \end{array}\right]$

15. True or false: a row of all zeros in an augmented matrix always signals *no solution*.

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16. Solve $\begin{cases} x + 2y - z = 3 \\ 2x + y + z = 4 \\ 3x - y + 2z = 5 \end{cases}$ by row reduction. ________

$$\underset{\textbf{augmented}}{\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & 4 \\ 3 & -1 & 2 & 5 \end{array}\right]} \Longrightarrow \underset{R_2 - 2R_1,\ R_3 - 3R_1}{\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -3 & 3 & -2 \\ 0 & -7 & 5 & -4 \end{array}\right]}$$

$$\underset{-\frac{1}{3}R_2}{\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & -7 & 5 & -4 \end{array}\right]} \Longrightarrow \underset{R_3 + 7R_2}{\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & -2 & \frac{2}{3} \end{array}\right]}$$

17. What does each elementary row operation correspond to, in terms of the original equations? ________

18. Solve $\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$ by row-reduction. ________

$$\underset{\textbf{start}}{\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & 2 & 6 \end{array}\right]} \Longrightarrow \underset{R_2 - 2R_1}{\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]}$$

19. Solve $\begin{cases} x + y + z = 2 \\ x + y + z = 5 \end{cases}$ by row-reduction. Classify. ________

$$\underset{\textbf{start}}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 5 \end{array}\right]} \Longrightarrow \underset{R_2 - R_1}{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]}$$

20. Solve $\begin{cases} x + y = 5 \\ 2x - 3y = 0 \end{cases}$ by row-reduction. ________

$$\underset{\textbf{start}}{\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & -3 & 0 \end{array}\right]} \Longrightarrow \underset{R_2 - 2R_1}{\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & -5 & -10 \end{array}\right]}$$

$$\underset{-\frac{1}{5}R_2}{\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \end{array}\right]} \Longrightarrow \underset{R_1 - R_2}{\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]}$$

Word Problems

21. A small business borrows from three banks at different rates. The total borrowed is $24,000. Bank A's rate is 5%, Bank B's is 6%, and Bank C's is 7%; the total interest for the year is $1,480. Bank C lent twice as much as Bank A. Use Gaussian elimination on the system to find how much each bank lent. ____

22. A school cafeteria sells three lunches priced at $4, $5, and $6. On Tuesday they sold 90 lunches for $445 in revenue. The number of $5 lunches equaled the combined number of $4 and $6 lunches. How many of each type sold? ____

23. A delivery service routes packages through three sorting centers. Center A handles 40% of the packages, Center B handles 35%, and Center C handles the rest. The total daily volume is $1{,}200$ packages. Use a matrix equation or Gaussian elimination to find how many packages each center processes. ____

24. A small business builds three product types A, B, and C, each needing time on two machines. One unit of A needs 1 hr on M1 and 2 hrs on M2; one unit of B needs 2 hrs on M1 and 1 hr on M2; one unit of C needs 1 hr on M1 and 1 hr on M2. M1 has 12 hours available and M2 has 14 hours. The company also wants to make 9 units total. Use Gaussian elimination to find the number of each. ____

Additional Practice

25. State the dimensions of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. ____

26. Add $\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix}$. ____

27. Subtract $\begin{bmatrix} 7 & 2 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$. ____

28. Find $\det \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$. ____

29. Find entry a_{21} in $\begin{bmatrix} 8 & 9 \\ -3 & 4 \end{bmatrix}$. ____

Answer Keys

1. $\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -4 & -5 \end{array}\right]$
2. multiply a row by 0
3. $\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -11 \end{array}\right]$
4. $(x, y) = (4, -3)$
5. no solution (inconsistent)
6. $(x, y) = (3, 1)$
7. $(x, y, z) = (1, 2, -1)$
8. infinitely many solutions
9. $(x, y) = (\frac{17}{5}, \frac{11}{5})$
10. $(x, y) = (3, 1)$
11. no solution
12. inconsistent: no solution
13. $(x, y, z) = (\frac{5}{2}, \frac{3}{2}, 2)$
14. infinitely many: y free
15. false
16. $(x, y, z) = (2, \frac{1}{3}, -\frac{1}{3})$
17. equation-level operations
18. infinitely many: y free
19. no solution (inconsistent)
20. $(x, y) = (3, 2)$
21. $A = \$4000,\ B = \$12{,}000,\ C = \$8000$
22. $\$4{:}\ 25,\ \$5{:}\ 45,\ \$6{:}\ 20$
23. $A = 480,\ B = 420,\ C = 300$
24. $A = 5,\ B = 3,\ C = 1$

Additional Practice Answers

25. 2×3
26. $\begin{bmatrix} 4 & 3 \\ 7 & 6 \end{bmatrix}$
27. $\begin{bmatrix} 4 & -2 \\ 3 & 4 \end{bmatrix}$
28. 2
29. -3

Additional Practice: Answers for all numbered items, including the added practice, are shown in the grid above.

Step-by-Step Explanations

1. Row 1: coefficients $2, 3$ and constant 7. Row 2: coefficients $1, -4$ and constant -5. The vertical bar marks where coefficients end and constants begin.

2. The three legal operations are swap, scale by a *nonzero* constant, and add a multiple of one row to another. Multiplying by 0 wipes out the row's information and can't be undone — not reversible, so not allowed.

3. Operate on row 2 only: $(3-3(1), -1-3(2), 4-3(5)) = (0, -7, -11)$. Row 1 stays put. (The operation $R_2 \to R_2 - 3R_1$ zeros out the first column of row 2, the opening move in Gaussian elimination.)

4. In RREF, leading 1s in columns for x and y mean the right column gives the variable values directly: $x = 4$, $y = -3$.

5. Read the row as an equation: $0x + 0y = 5$, which simplifies to $0 = 5$ — impossible. Whenever row reduction produces $[0\ 0\ \cdots\ 0 \mid k]$ with $k \neq 0$, the system is inconsistent.

6. Start with $[1, 2|5; 2, 5|11]$. $R_2 \to R_2 - 2R_1$ gives $[1, 2|5; 0, 1|1]$. Then $R_1 \to R_1 - 2R_2$ gives $[1, 0|3; 0, 1|1]$. RREF reads $x = 3$, $y = 1$. Verify: $3 + 2 = 5\ \checkmark, 6 + 5 = 11\ \checkmark$.

7. Back-substitute: row 3 gives $z = -1$. Row 2: $y - z = 3 \Rightarrow y + 1 = 3 \Rightarrow y = 2$. Row 1: $x + y - z = 4 \Rightarrow x + 2 + 1 = 4 \Rightarrow x = 1$. Check: $1 + 2 - (-1) = 4$, $2 - 2 + (-1) = -1, 1 + 4 + (-1) = 4$ all $\checkmark$.

8. The third row $0 = 0$ is redundant, not contradictory. Columns 1 and 2 have pivots, but column 3 (for z) has no pivot — z is free. With a free variable, infinitely many solutions: $x = 5 - 2z, y = 3 + z, z \in \mathbb{R}$.

9. From row 2: $y = \frac{11}{5}$. Row 1: $2x + y = 9 \Rightarrow 2x = 9 - \frac{11}{5} = \frac{34}{5} \Rightarrow x = \frac{17}{5}$. Verify: $2(\frac{17}{5}) + \frac{11}{5} = \frac{45}{5} = 9\ \checkmark; 4(\frac{17}{5}) - 3(\frac{11}{5}) = \frac{68-33}{5} = \frac{35}{5} = 7\ \checkmark$.

10. Swap rows first so the leading entry is 1 (cleaner pivots). Then clear below the pivot. Final RREF reads $x = 3, y = 1$. Verify: $3(3) + 2(1) = 11\ \checkmark$ and $3 - 1 = 2\ \checkmark$.

11. Row 2 says $0x + 0y = 5$, which is impossible. Inconsistent system; the two equations describe parallel lines.

12. Apply $R_1 - 2R_2$: $(3-2, 6-4, 9-8) = (1, 2, 1)$. Then $R_2 - R_1$: $(0, 0, 3)$. The row $[0, 0|3]$ is the contradiction $0 = 3$, so no solution. (Row reduction reveals the truth: these equations don't agree.)

13. Row 3 gives $z = 2$ directly. Row 2: $2y + z = 5 \Rightarrow 2y + 2 = 5 \Rightarrow y = \frac{3}{2}$. Row 1: $x + y + z = 6 \Rightarrow x + \frac{3}{2} + 2 = 6 \Rightarrow x = \frac{5}{2}$. So $(x, y, z) = (\frac{5}{2}, \frac{3}{2}, 2)$. Verify: $\frac{5}{2} + \frac{3}{2} + 2 = 6\ \checkmark, 2(\frac{3}{2}) + 2 = 5\ \checkmark, z = 2\ \checkmark$.

14. Row 2 becomes all zeros — a redundant equation, not a contradiction. The surviving equation is $x + \frac{3}{2}y = 4$, giving the solution set $x = 4 - \frac{3}{2}y$ with $y \in \mathbb{R}$. (Two equations, one line on the plane — infinitely many points.)

15. A row of *all* zeros (including the constant column) reads $0 = 0$ — always true, a redundant equation. The *no-solution* signal is $[0\ 0\ \cdots\ 0 \mid k]$ with $k \neq 0$ — a contradiction. Watch the right side: it makes the difference.

16. Row 3: $-2z = \frac{2}{3} \Rightarrow z = -\frac{1}{3}$. Row 2: $y - z = \frac{2}{3} \Rightarrow y = \frac{2}{3} + (-\frac{1}{3}) = \frac{1}{3}$. Row 1: $x + 2y - z = 3 \Rightarrow x + \frac{2}{3} - (-\frac{1}{3}) = 3 \Rightarrow x + 1 = 3 \Rightarrow x = 2$. So $(x, y, z) = (2, \frac{1}{3}, -\frac{1}{3})$. Verify the second equation: $2(2) + \frac{1}{3} + (-\frac{1}{3}) = 4\ \checkmark$.

17. Swapping rows = swapping the order of two equations. Scaling a row by $c \neq 0$ = multiplying that equation through by c. Adding a multiple of one row to another = adding a multiple of one equation to another. All preserve the solution set because they're standard algebra moves.

18. Row 2 becomes all zeros — equation 2 is just $2\times$ equation 1, so it adds no new information. Solution set: $x = 3 - y$, with y free. Geometrically, the two ëquationsäre the same line.

19. Row 2 becomes $[0, 0, 0|3]$ — the contradiction $0 = 3$. The two equations describe parallel planes; they never meet.

20. Three row operations cleaned this up. RREF reads $x = 3$, $y = 2$. Verify: $3 + 2 = 5\ \checkmark, 6 - 6 = 0\ \checkmark$.

21. Let a, b, c be the amounts (in thousands of dollars). System: $a + b + c = 24$, $0.05a + 0.06b + 0.07c = 1.48$, and $c = 2a$. Substitute $c = 2a$ into the first: $a + b + 2a = 24$, so $3a + b = 24$ and $b = 24 - 3a$. Plug into the interest equation (multiplied by 100): $5a + 6b + 7c = 148$, giving $5a + 6(24 - 3a) + 7(2a) = 148 \Rightarrow 5a + 144 - 18a + 14a = 148 \Rightarrow a = 4$. Then $b = 24 - 12 = 12$ and $c = 8$. Final: Bank A \$4000, Bank B \$12,000, Bank C \$8000. Verify total $24{,}000\ \checkmark$, interest $200 + 720 + 560 = 1480\ \checkmark$, and $c = 2a\ \checkmark$.

22. Let x, y, z be the counts at \$4, \$5, \$6. System: $x + y + z = 90$, $4x + 5y + 6z = 445$, and $y = x + z$. Substitute $y = x + z$ into the count equation: $2x + 2z = 90 \Rightarrow x + z = 45$, so $y = 45$. Plug $y = 45$ into the revenue equation: $4x + 225 + 6z = 445 \Rightarrow 4x + 6z = 220$. Combined with $x + z = 45$ (times 4 gives $4x + 4z = 180$), subtract: $2z = 40 \Rightarrow z = 20$, so $x = 25$. Counts: \$4 lunches $= 25$, \$5 lunches $= 45$, \$6 lunches $= 20$. Verify: $25 + 45 + 20 = 90\ \checkmark$; revenue $100 + 225 + 120 = 445\ \checkmark$; $y = 45 = 25 + 20\ \checkmark$.

23. Direct percentages: $A = 0.40(1200) = 480$, $B = 0.35(1200) = 420$, $C = 0.25(1200) = 300$. (As a matrix equation: with $a + b + c = 1200$ and the fixed proportions, the unique solution is integer. Verify: $480 + 420 + 300 = 1200\ \checkmark$.)

24. Let a, b, c be the units of A, B, C. System: $a + 2b + c = 12$ (M1), $2a + b + c = 14$ (M2), $a + b + c = 9$ (total). Subtract equation 3 from equation 1: $b = 3$. Subtract equation 3 from equation 2: $a = 5$. Then from equation 3: $c = 9 - 5 - 3 = 1$. Verify M1: $5 + 6 + 1 = 12\ \checkmark$, M2: $10 + 3 + 1 = 14\ \checkmark$, total $5 + 3 + 1 = 9\ \checkmark$. Clean integer answer.

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